# Incomplete correlation-sensitive preferences: An axiomatic framework for decision making under uncertainty

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## Abstract

This paper develops a unified axiomatic framework for decision making under uncertainty, from which both correlation-sensitive and expected multi-utility models emerge as special cases. Unlike standard correlation-sensitive models that assume completeness, it allows incomparability by replacing completeness with two natural axioms: reflexivity, requiring consistency under symmetric comparisons, and monotonicity, ensuring that mixtures with incomparable options cannot reverse existing preferences. When transitivity is additionally imposed, the framework collapses to the expected multi-utility model. The framework offers a foundation for understanding how incompleteness, correlation sensitivity, and transitivity jointly shape choice under uncertainty.

## 1 Introduction

In choices among uncertain alternatives, decision makers often care about how the outcomes of those alternatives are correlated, yet they may be unwilling or unable to form a complete ranking of all available options. This paper develops an axiomatic framework for preferences that reflect both sensitivity to correlation structures among outcomes and the possibility of incomparability between options.

The framework formalizes a simple but pervasive observation: when evaluating uncertain options, people do not assess them in isolation but care about how

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their outcomes are linked across states of the world; these interdependencies, in turn, make comparisons harder to resolve and can lead to indecisiveness. The model represents such behavior through multiple ranking systems that evaluate the same options, where disagreements among them generate partial rather than complete orderings. When transitivity is additionally imposed, the framework collapses to the expected multi-utility representation, which accommodates indecisiveness but not correlation sensitivity.

The paper's main contribution is to unify two strands of decision theory. Correlation-sensitive models capture how the joint distribution of outcomes, that is, the correlation structure among outcomes, influences choice but typically impose completeness[1]. Expected multi-utility models, by contrast, allow incompleteness but disregard correlations. This paper bridges these approaches by relaxing the completeness assumption in correlation-sensitive models and introducing Reflexivity and Monotonicity instead, thereby developing a framework in which both correlation-sensitive and expected multi-utility models emerge as special cases.

To see why correlation matters, consider an investor choosing between two startups. The first is relatively safe, with a modest return conditional on success and a 10% probability of success. The second is a "moonshot," offering a massive payoff but only a 5% chance of success. If the risks are independent, the expected payoff of the moonshot may look more attractive: since both ventures are highly likely to fail, the possibility of an extraordinary upside outweighs the modest gain of the safer choice. The left panel of Figure 1 illustrates this case under independent risks.

Now change only the correlation structure, so that the risks are correlated because both startups operate in the same sector, such as artificial intelligence, and face a common regulatory risk. With probability 0.9, a new regulation arrives and both ventures fail simultaneously. In that state, the choice of startup is irrelevant: whichever option the investor selects, the outcome is the same. A decision maker tends to give such states less weight because the counterfactual payoff would not have differed. Attention is therefore shifted to the remaining 10% of the time when regulation does not occur. In those states, the safer startup succeeds with certainty, while the moonshot succeeds with probability 0.5. The right panel of Figure 1 illustrates this case under correlated risks.

Notice that the marginal success probabilities of each startup remain un-

[1] Preferences are complete when, for any two alternatives, individuals are able to rank them; otherwise, they are incomplete.

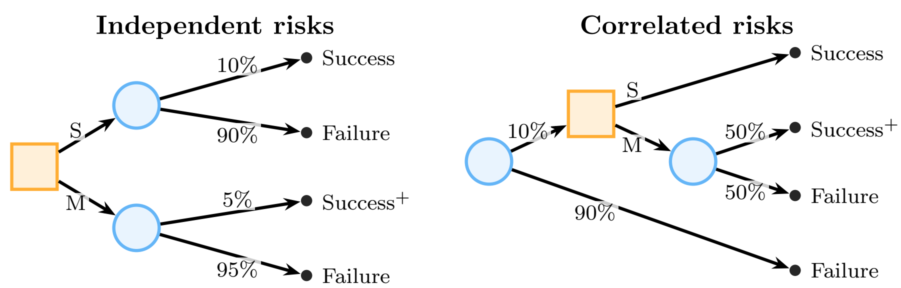


Figure 1: Independent versus correlated risks in the startup example. Decision nodes (squares) indicate points where the investor's choice between the safer startup (S) and the moonshot (M) is imposed, although the decision itself may have been made earlier. Chance nodes (circles) represent random events with the indicated probabilities. The left panel illustrates independent risks: the safer startup with a 10% probability of modest success and the "moonshot" with a 5% probability of a large payoff. The right panel illustrates correlated risks: both startups operate in the same sector and fail together with 90% probability if regulation occurs; when regulation does not occur (10% probability), the safer startup succeeds with certainty, while the moonshot succeeds with probability 50%. The choice is made before the realization of the regulatory risk, so the options and their marginal probabilities are identical in both panels. The marginal probabilities of each option remain unchanged, but the correlation structure changes counterfactual comparisons and may shift preferences toward the safer option.

changed: the safer one still succeeds with probability 0.10 overall, and the moonshot with probability 0.05. What has changed is how their outcomes are linked. When regulation does not occur, the safer venture succeeds with certainty, whereas the moonshot succeeds only half of the time. Under independent risks, the rare prospect of a large payoff makes the moonshot attractive despite its low overall probability. Under correlated risks, the states where both ventures fail together remove that attraction, and the safer option may become the more compelling choice.

This example illustrates the core notion of correlation sensitivity: a reversal or shift in preference rankings that arises solely from a change in the correlation structure of outcomes. A correlation-sensitive decision maker evaluates options not only by their marginal probabilities, but also by the counterfactual outcomes that would have materialized under alternative choices. When outcomes are linked, these counterfactual comparisons change, and with them the relative ranking of options, even though each option's individual risk profile remains fixed.

The startup example also highlights a second feature of decision-making: correlation sensitivity not only changes preferences but also increases the complexity of evaluation. When the joint probability distribution or correlation structure is irrelevant, each lottery can be assessed solely based on its marginal probabilities. When correlation becomes relevant, however, the decision maker must additionally consider how counterfactual payoffs co-move across states. This introduces contextual interdependencies that complicate the evaluation problem.

Additional interdependencies reflect greater contextual complexity and, with it, increased difficulty in resolving comparisons. A natural behavioral response to such contextual complexity is indecisiveness, or more precisely, the incompleteness of preferences. When the informational burden of fully processing correlations is too large, the decision maker may refrain from collapsing all comparisons into a complete ranking, leaving some alternatives incomparable.

This perspective links two strands of the literature. Classical models treat incompleteness as exogenous, arising from multiple evaluative criteria or indecisiveness. In contrast, the present framework views incompleteness as endogenous to the informational demands of correlation sensitivity. Environments with stronger dependence among outcomes entail richer interdependencies and therefore a greater scope for incomparability in preferences.

The model developed in this paper represents indecisiveness through multiple ranking systems that may yield conflicting evaluations, capturing the structural sources of incomparability. As correlation sensitivity increases the interdependence among outcomes, comparisons become more context dependent and harder to resolve. In such environments, individuals must assess alternatives along multiple and sometimes incommensurable dimensions, making incompleteness a natural byproduct rather than an anomaly. This perspective also provides a behavioral interpretation of empirical irregularities such as preference reversals, context effects, and framing anomalies; instead of reflecting noise or irrationality, these patterns may reveal genuine incompleteness in underlying preferences.

The correlation-sensitive representation is obtained by imposing Completeness, Strong Independence, and Continuity on preferences, Lanzani (2022). The representation for incomplete correlation-sensitive preferences is obtained by replacing the Completeness axiom with Reflexivity and Monotonicity. Hence, imposing Reflexivity, Monotonicity, Strong Independence, and Continuity on the preference set yields the representation. Continuity plays a mainly technical role, while Reflexivity ensures that when two options have identical marginal distributions and a symmetric correlation structure, the decision maker is indifferent between them.

Strong Independence extends the classic strong independence axiom to correlated settings. It requires that if $A \succsim B$ and $C \succsim D$, then for any $\alpha \in (0,1)$,

$$\alpha A + (1-\alpha)C \succsim \alpha B + (1-\alpha)D,$$

where the mixtures share the same probability $\alpha$. This distinction is crucial: under correlation sensitivity, the decision maker cares not only about the chosen option's outcome but also about the counterfactual outcome of having chosen differently. In this mixture interpretation, with probability $\alpha$ the comparison is between $A$ from the first mixture and $B$ from the second, and with probability $1-\alpha$ between $C$ and $D$. Both mixtures are therefore tied to a single probabilistic draw.

Monotonicity requires that preferences over mixtures preserve established rankings. If one option is preferred to another, then combining them with a pair of incomparable alternatives cannot reverse that ranking. Formally, if $A \succsim B$ and $C$ and $D$ are incomparable, then for any $\alpha \in (0,1)$,

$$\alpha B + (1-\alpha)D \not\succsim \alpha A + (1-\alpha)C,$$

with the mixtures again sharing the same probability $\alpha$. Thus, with probability $\alpha$ the comparison involves $A$ versus $B$, and with probability $1-\alpha$ it involves $C$ versus $D$. Monotonicity, therefore, rules out preference reversals induced by mixing with incomparable options, while still allowing incomparability to remain a feature of the relation.

As an example of the Monotonicity axiom, consider a DM deciding how to commute to work. She finds walking and biking incomparable; neither is strictly preferred to the other, but both are strictly preferred to driving, which in turn is strictly preferred to taking the subway: Walking$||$ Bike $\succ$ Car $\succsim$ Subway. On rainy days, walking and biking are unavailable, leaving only Car and Subway, for which the ranking is known: Car is preferred to Subway. On sunny days, all options are available, but walking and biking remain incomparable.

Now consider two commuting plans:

Plan 1: Subway on rainy days, Bike on sunny days.

Plan 2: Car on rainy days, Walk on sunny days.

Without Monotonicity, it could be possible that Plan 1 is ranked above Plan 2, even though on rainy days car is known to be strictly better than subway. Monotonicity rules this out: When options are mixed with incomparable alternatives (bike vs. walk), the known ranking between car and subway cannot be reversed. Hence, Plan 1 cannot be preferred to Plan 2.

Finally, note that if Walking were (weakly) preferred to Bike, then by Strong Independence Plan 1 could not be preferred to Plan 2 either.

If preferences satisfy Reflexivity, Monotonicity, Strong Independence, and Continuity, these conditions are both necessary and sufficient for the model representation. Let $X$ denote the set of all possible outcomes that options can yield in different states of the world, and let $x \in X$ and $y \in X$ represent the possible outcomes of options $A$ and $B$, respectively. Preferences that satisfy the above axioms admit the representation if there exists a nonempty subset $\Phi$ of the subspace of skew-symmetric functions[2] such that for every pair of options $A$ and $B$ in the choice set, $A$ is weakly preferred to $B$ if and only if

$$\sum_{x,y} \phi(x,y)\pi(x,y) \geqslant 0 \text{ for every } \phi \in \Phi.$$

Here, $\pi(x,y)$ denotes the probability of the state of the world in which option $A$ yields $x$ and option $B$ yields $y$. Intuitively, $\phi(x,y)$ captures how much the joint realization $(x,y)$ favors $x$ over $y$: $\phi(x,y) \geq 0$ if and only if $x$ is (weakly) preferred to $y$, and larger values indicate a stronger comparison in favor of $x$. The collection $\Phi$ can be interpreted as a set of ranking systems, each providing a separate evaluation of the options; when these rankings conflict, alternatives remain incomparable, giving rise to incompleteness.

As a special case, if $\Phi$ consists of a single $\phi$, unique up to a positive linear transformation, the preference is complete.Lanzani (2022) shows that Completeness, Strong Independence, and Archimedean Continuity are then necessary and sufficient conditions for such a correlation-sensitive representation. Moreover, if Transitivity is added, the model reduces to the standard Expected Utility model.

If Transitivity is imposed on the preference set in addition to Reflexivity,

[2] A function $\phi : X \times X \to \mathbb{R}$ is *skew-symmetric* if $\phi(x,y) = -\phi(y,x)$ for all $x, y \in X$. In particular, this implies $\phi(x,x) = 0$ for every $x \in X$.

Strong Independence, Monotonicity, and Continuity, the representation reduces to the Expected Multi-Utility model, Dubra et al. (2004).

The Expected Multi-Utility model establishes that a binary relation over lotteries satisfies Reflexivity, Transitivity, Independence, and Continuity if and only if there exists a closed and convex set of utility functions such that, for every pair of lotteries, the first is weakly preferred to the second if and only if its expected utility is greater than or equal to that of the second for all functions in the set. Having multiple utility functions in the representation can be interpreted as the decision maker being uncertain about, or unwilling to commit to, a single evaluative stance. For example, a subject may simultaneously have different risk attitudes, with none being strictly more valid than the others. In such cases, distinct utility functions provide competing rankings over risky options, and incompleteness arises whenever these rankings disagree. This framework captures the intuition that individuals may find it difficult to make definitive choices; however, it does not account for correlation sensitivity.

As mentioned before, the need to accommodate incompleteness becomes especially important in correlation-sensitive environments. When evaluations depend not only on marginal distributions but also on the correlation structure of outcomes, disagreements between evaluative perspectives are more likely to arise and harder to resolve. A single complete ordering risks obscuring these conflicts through arbitrary tie-breaking or ad hoc assumptions. By relaxing completeness, the model offers a richer and more accurate account of choice behavior in such settings. It emphasizes that indecisiveness can be a systematic feature of rational evaluation when correlation-sensitive concerns increase the complexity of the decision problem.

The next section briefly reviews the related literature. The paper is organized as follows. Section 2 introduces the notion of preference sets, following the framework developed by Fishburn (1990a), which provides the foundation for incorporating correlation structures into the notation. Section 3 discusses the interpretation of mixtures in this framework. Section 4 establishes the characterization of the incomplete correlation-sensitive representation. Section 5 discusses the role of transitivity and presents the second main result, showing that the model reduces to the expected multi-utility framework when transitivity is imposed. Section 6 concludes.

## 1.1 Related literature

Correlation-sensitive models of decision making under uncertainty account for how the correlation between risky options can influence choice behavior (Bell (1982), Fishburn (1989), and Kőszegi and Szeidl (2013)). Frameworks such as regret theory (Loomes and Sugden, 1982) and salience theory (Bordalo et al., 2012) fall within this class. In regret theory, the effect of correlation structure arises from counterfactual comparisons between the chosen option and the foregone alternative, whereas in salience theory, it is driven by the allocation of attention and weight to different possible states.

The foundations of this literature trace back to the seminal contributions of Fishburn (1989), Sugden (1993), and Quiggin (1994). Subsequent work has provided formal axiomatizations of these models: Diecidue and Somasundaram (2017) for regret theory (Loomes and Sugden, 1982), and Ellis and Masatlioglu (2022) together with Lanzani (2022) for rank-dependent and continuous versions of salience theory, respectively. Furthermore, Herweg and Müller (2021) demonstrates that regret theory is a special case of salience theory, while salience itself can be viewed as a special case of generalized regret theory.

To incorporate the correlation structure into the notation, I build on the preference set framework introduced by Fishburn (1990a) (application in SSA model Fishburn (1990b)) and employed by Lanzani (2022) to generalize correlation-sensitive models. Rather than defining a binary relation over lotteries, the preference set approach defines a binary relation over options, considering their joint distribution. where each option is characterized not only by its marginal probability distribution but also by its joint distribution with every other option.

The correlation-sensitive model (Lanzani, 2022) is a special case of the model representation developed in this paper. Moreover, when Transitivity is imposed in addition to Reflexivity, Strong Independence, Monotonicity, and Continuity, the representation reduces to the Expected Multi-Utility model (Ok et al. (2002) and Dubra et al. (2004)).

Dubra et al. (2004) show that a binary relation over lotteries is a preference relation satisfying Independence and Continuity if and only if there exists a closed and convex set of utility functions such that, for every pair of lotteries, the first is weakly preferred to the second if and only if its expected utility is greater than or equal to that of the second for all functions in the set. In this framework, a preference relation is taken to be a reflexive and transitive binary relation, in contrast to the standard theory where completeness is also assumed.

In the expected multi-utility model of Dubra et al. (2004), preferences can be represented by a closed and convex set of utility functions, each reflecting a possible evaluation of lotteries. Having multiple utility functions in the representation can be interpreted as the decision maker being uncertain about, or unwilling to commit to, a single evaluative stance. For example, a subject may simultaneously have different risk attitudes, with none being strictly more valid than the others. In such cases, distinct utility functions provide competing rankings over risky options, and incompleteness arises whenever these rankings disagree. This perspective captures the intuition that individuals often experience difficulty making definitive choices when options involve trade-offs across different dimensions.

## 2 Preference sets

The preference set framework of Fishburn (1990a), is adopted to generalize correlation-sensitive models. In this framework, preferences are defined over options characterized by both their marginal and joint distributions, rather than solely over lotteries.

Let $X = \{x_1, x_2, x_3, \ldots, x_n\}$ denote a fixed set of possible outcomes, and let $\Delta(X \times X)$ be the set of all joint probability distributions over $X \times X$. Each element of $\Delta(X \times X)$ specifies the joint distribution of payoffs associated with a pair of options.

$\pi \in \Delta(X \times X)$:

$$\begin{array}{cc} & \begin{array}{ccccc} x_1 & x_2 & x_3 & \cdots & x_n \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{array} & \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} & \cdots & \pi_{1n} \\ \pi_{21} & \pi_{22} & \pi_{23} & \cdots & \pi_{2n} \\ \pi_{31} & \pi_{32} & \pi_{33} & \cdots & \pi_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \pi_{n1} & \pi_{n2} & \pi_{n3} & \cdots & \pi_{nn} \end{bmatrix} \end{array}$$

$\pi_{ij}$ denotes the probability that the realized state yields outcome $x_i$ from the first option (listed in the rows) and outcome $x_j$ from the second option (listed in the columns).

The joint probability distributions corresponding to the cases in Figure 1 are shown below. In each matrix, the safer startup $S$ is represented by the rows. From left to right, the matrices correspond to the independent and correlated cases, respectively.

$$\begin{array}{c} \\ \text{Failure} \\ \text{Success} \\ \text{Success}^+ \end{array} \begin{array}{c} \begin{array}{ccc} \text{Failure} & \text{Success} & \text{Success}^+ \end{array} \\ \begin{bmatrix} 0 & 0.005 & 0.095 \\ 0 & 0 & 0 \\ 0 & 0.045 & 0.855 \end{bmatrix} \end{array} \quad \begin{array}{c} \\ \text{Failure} \\ \text{Success} \\ \text{Success}^+ \end{array} \begin{array}{c} \begin{array}{ccc} \text{Failure} & \text{Success} & \text{Success}^+ \end{array} \\ \begin{bmatrix} 0 & 0.05 & 0.05 \\ 0 & 0 & 0 \\ 0 & 0 & 0.9 \end{bmatrix} \end{array}$$

Preferences are represented by a preference set $\Pi \subseteq \Delta(X \times X)$. The interpretation is that, for every joint distribution $\pi$ over $X \times X$, the DM prefers to receive the outcome indicated by the row.

It is important to note that, in this framework, the state space is endogenous: it is constructed relative to the particular pair of options under consideration. For example, when acts are defined over a state space with objective probabilities, any pair of acts can be associated with a joint distribution $\pi \in \Delta(X \times X)$. However, if there are more than two acts, knowing all pairwise joint distributions does not, in general, identify a unique set of objective probabilities over the underlying states. To illustrate, suppose there are three acts and four outcomes. The maximum number of possible states is $4^3$, implying $4^3$ unknown state probabilities. From the pairwise joint distributions, there are $3 \times 4^2$ equations, insufficient to uniquely determine the probabilities. Consequently, some information about the higher-order correlation structure across all options is inevitably lost when more than two options are present. In this sense, the preference set framework is less expressive than a binary relation over acts, yet more structured than a binary relation over lotteries.

## 3 Mixtures

The correlation structure between two risky options refers to the statistical dependence of their outcomes. Formally, it is determined by the joint probability distribution, which specifies how the realization of one option is linked to the realization of the other. Two options may share identical marginal distributions yet differ in how their outcomes co-move. A decision maker may evaluate the same lotteries differently depending on whether they are positively, negatively, or independently correlated, even though their individual outcome probabilities remain unchanged. Sensitivity to the correlation structure is a fundamental feature of the model. I capture this by letting $\pi \in \Delta(X \times X)$ denote the joint probability distribution over the outcomes of a pair of options.

Incorporating correlation sensitivity explicitly into the axiomatic framework, particularly in the definition of mixtures, is essential to ensure that the representation faithfully reflects how correlations influence choice behavior.

One of the central axioms in decision theory is the Independence axiom. It states that if option $A$ is weakly preferred to option $B$, then for any other option $C$ and any $\alpha \in (0,1)$, the mixture $\alpha A + (1-\alpha)C$ is weakly preferred to $\alpha B + (1-\alpha)C$:

$$A \succsim B, \alpha \in (0,1) \Rightarrow \alpha A + (1-\alpha)C \succsim \alpha B + (1-\alpha)C.$$

The usual interpretation is that in the mixture, the decision maker receives option $A$ (or $B$) with probability $\alpha$, and option $C$ with probability $1-\alpha$. However, the axiom leaves implicit the correlation structure between the two mixture lotteries. In other words, when comparing $\alpha A + (1-\alpha)C$ and $\alpha B + (1-\alpha)C$, it is not specified how the randomization is jointly implemented, and thus the axiom requires the preference to hold regardless of the correlation structure between the two mixtures.

To make this point precise, let $X = \{A, B, C\}$ denote a set of possible outcomes (not necessarily payoffs), and consider $\Delta(X \times X)$, the set of all joint probability distributions over pairs of outcomes. Each element of $\Delta(X \times X)$ represents a possible correlation structure between the two mixtures. For example, the matrices below illustrate the cases of independence and perfect correlation for two mixtures $\alpha A + (1-\alpha)C$ and $\alpha B + (1-\alpha)C$ (with $\alpha A + (1-\alpha)C$ on the rows and $\alpha B + (1-\alpha)C$ on the columns). In the independent case (left matrix), knowing the outcome of one option provides no information about the outcome of the other. By contrast, in the perfectly correlated case (right matrix), once the outcome of one mixture is realized, the outcome of the other mixture is also fully determined.

$$\begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & \alpha^2 & \alpha(1-\alpha) \\ 0 & 0 & 0 \\ 0 & \alpha(1-\alpha) & (1-\alpha)^2 \end{bmatrix} \end{array} \qquad \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-\alpha \end{bmatrix} \end{array}$$

In the classic independence axiom, these are not the only cases; for any $r \in [0, \min(\alpha, 1-\alpha)]$, the matrix below represents a possible correlation structure between the two mixtures $\alpha A + (1-\alpha)C$ and $\alpha B + (1-\alpha)C$.

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & \alpha - r & r \\ 0 & 0 & 0 \\ 0 & r & 1-\alpha-r \end{bmatrix} \end{array}$$

The strong independence axiom is very similar, but instead of having the same option $C$ in both mixtures, I assume $C$ is weakly preferred to option $D$. Then, for any $\alpha \in (0,1)$, the mixture $\alpha A + (1-\alpha)C$ is weakly preferred to $\alpha B + (1-\alpha)D$:

$$A \succsim B, C \succsim D, \alpha \in (0,1) \Rightarrow \alpha A + (1-\alpha)C \succsim \alpha B + (1-\alpha)D.$$

As before, I can represent both the independent and the correlated cases (with rows and columns corresponding to zero probabilities omitted).

$$\begin{array}{cc} & \begin{array}{cc} B & D \end{array} \\ \begin{array}{c} A \\ C \end{array} & \begin{bmatrix} \alpha^2 & \alpha(1-\alpha) \\ \alpha(1-\alpha) & (1-\alpha)^2 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} B & D \end{array} \\ \begin{array}{c} A \\ C \end{array} & \begin{bmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{bmatrix} \end{array}$$

The importance of the correlation structure for a correlation-sensitive DM is that, in the correlated version, option $A$ is always compared to option $B$ and option $C$ is always compared to option $D$. This contrasts with the independent case, where $A$ or $B$ may instead be paired with $C$ or $D$, making the evaluation sensitive to the specification of the joint distribution.

In classical decision theories, the DM is typically assumed to be correlation-insensitive: the correlation structure is disregarded both in the specification of the choice set and in the interpretation of mixtures. By contrast, once the joint distribution is made explicit, the interpretation of mixtures becomes fundamentally different. Let $X$ be a set of outcomes, and let $\Delta(X \times X)$ denote the set of all joint distributions over $X \times X$. Any $\pi \in \Delta(X \times X)$ can be represented as a binary choice set, where the row marginal corresponds to the first option, the column marginal to the second option, and $\pi$ itself specifies their joint distribution. Now consider $\pi, \pi' \in \Delta(X \times X)$ and $\alpha \in (0,1)$. Since $\Delta(X \times X)$ is convex, the mixture $\alpha\pi + (1-\alpha)\pi'$ also belongs to $\Delta(X \times X)$. If $\pi$ encodes options $A$ (rows) and $B$ (columns), while $\pi'$ encodes options $C$ (rows) and $D$ (columns), then $\alpha\pi + (1-\alpha)\pi'$ corresponds to a correlated representation of the mixtures

$\alpha A + (1-\alpha)C$ and $\alpha B + (1-\alpha)D$.

$$\begin{array}{cc} & \begin{array}{cc} B & D \end{array} \\ \begin{array}{c} A \\ C \end{array} & \begin{bmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{bmatrix} \end{array}$$

As a reminder, $\pi \in \Pi$ indicates that the option represented on the row is weakly preferred to the option represented on the column. Within this framework, the strong independence axiom can be reformulated as follows:

$$\forall \pi, \pi' \in \Pi, \alpha \in (0,1) \Rightarrow \alpha\pi + (1-\alpha)\pi' \in \Pi.$$

That is, if two joint distributions $\pi$ and $\pi'$ both encode weak preference of the row option over the column option, then any convex combination of the two must preserve this weak preference relation.

In this framework, strong independence applies only to correlated mixtures of options. In this sense, it is a weaker condition than the classical strong independence axiom, which requires preservation of preference under mixtures with all possible correlation structures. In fact, the axiom here resembles a statewise dominance condition: because mixtures are correlated, each pair of options is evaluated state by state, so any mixture that is weakly preferred in every state will dominate its alternative.

To wrap up this section, it is important to note that mixtures in this setting represent correlated mixtures. Specifically, for any $\pi, \pi' \in \Delta(X \times X)$ and $\alpha \in (0,1)$, the mixture $\alpha\pi + (1-\alpha)\pi'$ corresponds to a situation where, with probability $\alpha$, the row and column options of $\pi$ are compared against each other, and with probability $1-\alpha$, those of $\pi'$ are compared. Consequently, when axioms are applied to such mixtures, they are inherently weaker than axioms requiring preservation of preference under all possible correlation structures.

# 4 Incomplete Correlation-sensitive Model

Let $X$ be an arbitrary nonempty finite set with $n$ elements, $|X| = n$. Elements of $X$ can represent outcomes, options, or probability distributions. The space of all possible joint correlation structures over $X \times X$ is denoted by $\Delta(X \times X)$, which I refer to simply as $\Delta$. Formally,

$$\Delta \coloneqq \left\{ \pi \in \mathbb{R}^{n\times n} \mid \pi_{ij} \geq 0 \text{ for all } i,j = 1,2,\ldots,n; \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_{ij} = 1 \right\}$$

Each $\pi \in \Delta$ represents a joint probability distribution over $X \times X$, where $\pi_{ij}$ denotes the probability assigned to the pair $(x_i, x_j) \in X \times X$. The set $\Delta$ is a closed, convex subset of $\mathbb{R}^{n^2}$ — specifically, the $(n^2-1)$-dimensional simplex within the hypercube $[0,1]^{n^2}$.

I define a subset $\Pi \subseteq \Delta$ to represent the decision-maker's (DM's) preference set. The interpretation is as follows: the DM faces a joint distribution $\pi \in \Delta$, and must choose whether to be paid according to the outcome indexed by the row or by the column.

$\pi \in \Pi$ if and only if the DM (weakly) prefers to be paid according to the row outcome rather than the column. That is,

$$\pi \in \Pi \Leftrightarrow \text{row outcome is (weakly) preferred.}$$

For any $\pi \in \Delta$, $\bar{\pi}$ is the transpose of $\pi$:

$$\forall (x,y) \in X \times X : \bar{\pi}(x,y) = \pi(y,x)$$

It simply relabels the row and column into each other. I define $\bar{\Pi}$ as the set of transposes of matrices in $\Pi$:

$$\bar{\Pi} \coloneqq \{\pi \in \Delta : \bar{\pi} \in \Pi\}.$$

If a distribution $\pi \in \Delta$ satisfies $\pi \notin \Pi$ and $\pi \notin \bar{\Pi}$, then the DM finds the row and column options incomparable, indicating an incomplete preference. I define the set of such distributions as:

$$I \coloneqq \Delta \backslash (\Pi \cup \bar{\Pi}).$$

I identify $C(X \times X)$ with the space of all real-valued functions on $X \times X$, i.e., $\mathbb{R}^{n\times n}$. The subspace of skew-symmetric functions corresponds to skew-symmetric matrices:

$$C^{ss}(X \times X) \coloneqq \{c \in C(X \times X) : c(x,y) = -c(y,x) \text{ for all } x,y \in X\}.$$

**Incomplete correlation-sensitive model representation:**
A preference set $\Pi$ admits an incomplete correlation-sensitive representation if there exists a nonempty subset $\Phi$ of $C^{ss}(X \times X)$ such that, for any $\pi$ in $\Delta(X \times X)$, I have $\pi \in \Pi$ iff

$$\sum_{x,y} \phi(x,y)\pi(x,y) \geqslant 0 \text{ for every } \phi \in \Phi.$$

In this case, I say that $\Phi$ is an incomplete correlation-sensitive representation for $\Pi$.

**Axiom 1.** *(Reflexivity) For any $\pi$ such that $\pi = \bar{\pi}$, both $\pi$ and $\bar{\pi}$ are in the preference set, $\pi, \bar{\pi} \in \Pi$.*

This axiom captures the idea that the DM should be indifferent between two options whenever the two share the same marginal distributions and the correlation structure is symmetric. Formally, Reflexivity guarantees that if $\pi$ is equal to its transpose, then both $\pi$ and $\bar{\pi}$ belong to the preference set, reflecting the principle that only differences in marginal probabilities or asymmetric correlation structures can justify a strict ranking or incomparability. In particular, symmetry cannot give rise to incomparability: the decision maker must treat such options as equivalent.

**Axiom 2.** *(Strong Independence) For all $\pi, \pi' \in \Pi$, and all $\alpha \in (0,1)$,*

$$\alpha\pi + (1-\alpha)\pi' \in \Pi.$$

*Moreover, if $\pi' \in \hat{\Pi}$, then*

$$\alpha\pi + (1-\alpha)\pi' \in \hat{\Pi}.$$

*$\hat{\Pi}$ is the strict preference set, $\hat{\Pi} = \{\pi \in \Pi : \bar{\pi} \notin \Pi\}$.*

As discussed in Section 3, the key difference between the classical strong independence axiom and its formulation in this setting is that I restrict attention to correlated mixtures. This restriction makes the axiom weaker than the classical version, which requires independence to hold across all possible correlation structures. In the present framework, strong independence simply ensures that the preference set $\Pi$ is convex.

**Axiom 3.** *(Monotonicity) If* $\pi \in \Pi$ *and* $\pi' \in I$*, then for all* $\alpha \in (0, 1)$*,*

$$\alpha\pi + (1 - \alpha)\pi' \notin \bar{\Pi}.$$

This axiom requires that correlated mixtures involving incomparable options cannot reverse the order between comparable ones. Formally, if $\pi \in \Pi$ and $\pi' \in I$ (incomparable), then any correlated mixture of the two options in $\pi$ with those in $\pi'$ must not generate the opposite ranking of $\pi$. In other words, preferences over mixtures must respect known preferences and cannot allow incomparability to override them.

Immediate results of this axiom:

If $\pi \in \bar{\Pi}$ and $\pi' \in I$, then for all $\alpha \in (0, 1)$,

$$\alpha\pi + (1 - \alpha)\pi' \notin \Pi$$

If $\pi \in (\Pi \cap \bar{\Pi})$ and $\pi' \in I$, then for all $\alpha \in (0, 1)$,

$$\alpha\pi + (1 - \alpha)\pi' \in I$$

**Axiom 4.** *(Continuity)* $\Pi$ *is closed.*

**Theorem 1.** *A preference set* $\Pi$ *satisfies axioms 1-4 if and only if* $\Pi$ *admits an incomplete correlation-sensitive representation.*

*Proof.* See Appendix A.1. ■

# 5 Transitivity

Defining transitivity in this framework is more subtle than in the classical case. In standard models, preferences are defined directly over probability distributions, and transitivity simply requires that if one option is preferred to a second and the second is preferred to a third, then the first must also be preferred to the third. In our setting, however, each option is not only represented by its own probability distribution but also by the correlation structure it admits with every other option. This means that the preference relation is inherently pairwise: the comparison between two options depends on the joint distribution that links them. As a result, formulating transitivity requires care, since the indirect comparison of two options through a third might rely on different correlation structures than their direct comparison. A suitable version of transitivity in this environment must therefore ensure that the preference relation remains logically coherent across chains of pairwise comparisons, while acknowledging that correlations are part of the primitives of choice rather than external assumptions.

Lanzani (2022) formulates transitivity with respect to the marginal probability distributions of the options only:
For all $\pi, \chi, \rho \in \Delta(X \times X)$, if $\pi_2 = \chi_1$, $\rho_1 = \pi_1$, and $\rho_2 = \chi_2$, then

$$(\pi \in \Pi, \chi \in \Pi) \Rightarrow \rho \in \Pi.$$

For any joint probability distribution $\pi \in \Delta(X \times X)$, $\pi_1, \pi_2 \in \Delta(X)$ are denoted as its first (row) and second (column) marginal distributions, respectively. This formulation avoids the complications of correlation structures by abstracting away from the joint distributions and focusing solely on the marginals.

When I move to the case of more than two options in our framework, some part of the information on the correlation structure among all the options is necessarily missing. In particular, while pairwise comparisons are grounded in well-defined joint distributions, extending these to a consistent ranking over three or more options requires compatibility conditions across the different pairwise correlation structures. The challenge is therefore to define a transitivity axiom that ensures coherence of the preference relation without assuming a complete specification of the higher-order joint distribution among all the options.

As an illustration, consider the matrices $\pi, \chi$, and $\rho$ in Table 1. These distributions satisfy the requirements of the transitivity axiom, namely $\pi_2 = \chi_1$, $\rho_1 = \pi_1$, and $\rho_2 = \chi_2$. Hence, if both $\pi$ and $\chi$ belong to the preference set, transitivity requires that $\rho$ must also belong to the preference set. In Lanzani (2022), this example is presented as a transitivity failure, and they argue that from the perspective of a correlation-sensitive DM, it is not plausible to have all $\pi$, $\chi$, and $\rho$ in $\Pi$ simultaneously. In contrast, I examine whether these options can be embedded in a common state space and show that their pairwise correlation structures are mutually inconsistent, making it impossible to construct a single joint probability space containing all three lotteries.

In Table 1, suppose that when three options are present in the choice set, the marginal distributions align as follows: the first option's marginal is $\rho_1 = \pi_1$, the second option's marginal is $\pi_2 = \chi_1$, and the third option's marginal is $\rho_2 = \chi_2$. Consider $\rho$: in the state where option 1 yields outcome 10 and option 3 yields

| $\pi$ | 7 | 2 |
|---|---|---|
| 10 | 0 | $\frac{1}{4}$ |
| 5 | $\frac{1}{2}$ | 0 |
| 0 | 0 | $\frac{1}{4}$ |

| $\chi$ | 8 | 1 |
|---|---|---|
| 7 | $\frac{1}{2}$ | 0 |
| 2 | 0 | $\frac{1}{2}$ |

| $\rho$ | 8 | 1 |
|---|---|---|
| 10 | $\frac{1}{4}$ | 0 |
| 5 | 0 | $\frac{1}{2}$ |
| 0 | $\frac{1}{4}$ | 0 |

Table 1: These three joint probability distributions illustrate a failure of Transitivity due to salience sensitivity (Lanzani, 2022). They argue that for a salience-sensitive DM, it is reasonable to have $\pi \in \Pi$, $\chi \in \Pi$, and $\rho \notin \Pi$.

outcome 8 (with probability $\frac{1}{4}$), there are two possible outcomes for option 2, namely 2 or 7. However, if option 2's outcome is 7, then the probability that option 1 yields 10 is zero; conversely, if option 2's outcome is 2, then the probability that option 3 yields 8 is zero. Hence, it is impossible to construct three distinct options consistent with these joint correlations. Put differently, there is no way to define a state space and associated probabilities that simultaneously realize all three options with the specified pairwise correlation structures.

With the notation of joint probability distributions for pairs of options, the correlation structure among all options is ignored when more than two options are under consideration. In the transitivity axiom of Lanzani (2022), the pairwise correlation structure is likewise ignored, which might lead to the nonexistence of a common state space. This axiom might be considered strong in a setting where the main assumption is that individuals are correlation-sensitive. Next, I formalize a correlation-consistent transitivity axiom that uses a triple joint distribution to ensure that pairwise correlation structures are compatible.

**Axiom 5.** *(Strong Transitivity) Let $\pi_{ABC} \in \Delta(X^3)$ be a joint distribution over three options, and denote its pairwise marginals by*

$$\pi_{AB}(x, y) = \sum_{z \in X} \pi_{ABC}(x, y, z),$$

$$\pi_{BC}(y, z) = \sum_{x \in X} \pi_{ABC}(x, y, z),$$

$$\pi_{AC}(x, z) = \sum_{y \in X} \pi_{ABC}(x, y, z).$$

*If $\pi_{AB} \in \Pi$ and $\pi_{BC} \in \Pi$, then $\pi_{AC} \in \Pi$.*

**Theorem 2.** *Let $\Pi$ admit an incomplete correlation-sensitive representation, i.e., let $\Phi$ be a nonempty subset of $C^{ss}(X \times X)$ representing $\Pi$. Then, the* *following statements are equivalent:*

1. $\Pi$ *satisfies Axiom 5;*
2. *For every* $\phi \in \Phi$*, it holds that*

$$\phi(x,z) = \phi(x,y) + \phi(y,z), \quad \forall x,y,z \in X.$$

*Proof.* See Appendix A.2. ∎

The new transitivity axiom explicitly requires the existence of a common state space, ruling out transitivity claims built on inconsistent pairwise marginals. In the literature, regret aversion is typically captured by $\phi(x,z)$ being greater than $\phi(x,y) + \phi(y,z)$ for all $x > y > z$. By Theorem 2, imposing transitivity forces the $\phi$ functions to be regret-neutral, that is, $\phi(x,z) = \phi(x,y) + \phi(y,z)$ for all $x, y, z \in X$. Consequently, models such as salience and regret, which are generally nontransitive, inherently assume regret aversion.

Regret neutrality, expressed as $\phi(x,z) = \phi(x,y) + \phi(y,z)$, together with the skew-symmetry of $\phi$, implies that $\phi(x,y)$ must take the separable form $\phi(x,y) = g(x) - g(y)$ for some function $g$.

Consider a function $\phi$ satisfying skew-symmetry, $\phi(x,y) = -\phi(y,x)$, and regret neutrality, $\phi(x,z) = \phi(x,y) + \phi(y,z)$. Let

$$\phi_1(x,y) = \frac{\partial \phi(x,y)}{\partial x}, \phi_2(x,y) = \frac{\partial \phi(x,y)}{\partial y}.$$

By regret neutrality,

$$\phi(x,z) = \phi(x,y) + \phi(y,z) \Rightarrow \phi_1(x,z) = \phi_1(x,y).$$

Differentiating with respect to $z$ gives

$$\frac{\partial \phi_1(x,z)}{\partial z} = \frac{\partial \phi_1(x,y)}{\partial z} \Rightarrow \phi_{12}(x,z) = 0.$$

Ruling out non-additive interactions, this implies that $\phi$ is additive in its arguments, confirming that the general form of $\phi$ consistent with skew-symmetry and regret neutrality is

$$\phi(x,y) = g(x) - g(y)$$

for some function $g$. Thus, imposing transitivity reduces $\phi$ to a separable form, making the model equivalent to the expected utility representation. In the context of incomplete preferences, the resulting structure corresponds to an expected multi-utility representation.

Since the framework allows for incomplete preferences, the transitivity axiom can be relaxed to a weaker requirement that merely prevents preference cycles. The Weak Transitivity axiom does not impose any preference among the options in a chain, but only rules out configurations that would result in cyclical rankings.

**Axiom 6.** *(Weak Transitivity) Let $\pi_{ABC} \in \Delta(X^3)$ be a joint distribution over three options, and denote its pairwise marginals by*

$$\pi_{AB}(x,y) = \sum_{z\in X} \pi_{ABC}(x,y,z),$$

$$\pi_{BC}(y,z) = \sum_{x\in X} \pi_{ABC}(x,y,z),$$

$$\pi_{AC}(x,z) = \sum_{y\in X} \pi_{ABC}(x,y,z).$$

*If $\pi_{AB} \in \Pi$ and $\pi_{BC} \in \Pi$, then $\bar{\pi}_{AC} \notin \hat{\Pi}$.*

**Conjecture 3** (Weak Transitivity and Sign-Diversity)**.** *Let $\Pi$ admit a correlation-sensitive multi-utility representation via a nonempty set $\Phi \subseteq \mathrm{Css}(X \times X)$, i.e.,*

$$\pi \in \Pi \quad \Longleftrightarrow \quad \sum_{x,y\in X} \phi(x,y)\,\pi(x,y) \;\geq 0 \quad \textit{for every } \phi \in \Phi.$$

*For $x, y, z \in X$, define the* triple gap

$$\Delta_\phi(x,y,z) \;:=\; \phi(x,z) \;-\; \phi(x,y) \;-\; \phi(y,z).$$

*Say that $\phi$ is regret-averse on $(x,y,z)$ if $\Delta_\phi(x,y,z) > 0$, regret-taking if $\Delta_\phi(x,y,z) < 0$, and regret-neutral if $\Delta_\phi(x,y,z) = 0$.*

*Then $\Pi$ satisfies* Weak Transitivity *if and only if, for every triple of distinct outcomes $x, y, z \in X$, one of the following holds:*

1. *There exist $\phi^+, \phi^- \in \Phi$ such that $\Delta_{\phi^+}(x,y,z) > 0$ and $\Delta_{\phi^-}(x,y,z) < 0$ (sign-diversity), or*

2. *$\Delta_\phi(x,y,z) = 0$ for all $\phi \in \Phi$ (neutrality).*

# 6 Conclusion

One of the key advantages of this representation is its capacity to accommodate incomplete preferences through the use of multiple evaluation parameters. When a single evaluation function $\phi$ suffices, the model recovers complete preferences. However, when multiple — and potentially conflicting — evaluations are required, incompleteness naturally emerges within the preference relation.

In the presence of incompleteness, external factors such as framing, representation, or cognitive biases may influence individuals and nudge them toward one option over another. This highlights that some individuals do not possess well-defined rankings for all alternatives — and, crucially, their choices need not be interpreted as arising from some hidden or fundamental decision-making criterion.

Allowing for explicit incompleteness addresses broader challenges in decision theory. It acknowledges that the absence of complete, transitive rankings is not necessarily a sign of irrationality, but rather an inherent feature of decision-making under complexity, uncertainty, or ambiguity. This framework challenges the traditional assumption of premature completeness and underscores the importance of models that can accommodate both temporary indecision and persistent incompleteness as stable outcomes.

The incomplete correlation-sensitive representation achieves this through a set of intuitive axioms. Because the framework recognizes correlation structure, mixtures are correlated. The strong independence axiom resembles a statewise dominance (or monotonicity) principle, stating that if one option yields weakly preferred outcomes in every state, it is weakly preferred overall. Another axiom governs mixtures involving comparable and incomparable options, ensuring that incomparable pairs cannot reverse the established order of known preferences through mixing. Together with reflexivity and continuity, these axioms characterize the incomplete correlation-sensitive representation.

When transitivity is imposed, the correlation sensitivity is removed, while the possibility of incompleteness remains. Both the complete correlation-sensitive model and the expected multi-utility model arise as special cases of the incomplete correlation-sensitive framework.

As the next step, a natural direction is to explore how the representation evolves under further relaxations of the axioms. For instance, replacing the strong independence axiom with a correlated independence condition may reveal how these axioms interact—particularly since the standard expected multi-utility model relies on the independence rather than its strong version. Similarly, removing the reflexivity axiom could provide insights into choice behavior when self-consistency is not guaranteed, allowing the model to explore decision contexts where preferences are formed dynamically or remain unresolved. These extensions would further clarify how the mixture-based assumptions shape the boundaries between correlation sensitivity, incompleteness, and classical multi-utility representations.

## References


Giacomo Lanzani. Correlation made simple: Applications to salience and regret theory. *The Quarterly Journal of Economics*, 137(2):959–987, 2022.

Juan Dubra, Fabio Maccheroni, and Efe A Ok. Expected utility theory without the completeness axiom. *Journal of Economic Theory*, 115(1):118–133, 2004.

Peter C Fishburn. Continuous nontransitive additive conjoint measurement. *Mathematical Social Sciences*, 20(2):165–193, 1990a.

David E Bell. Regret in decision making under uncertainty. *Operations research*, 30(5):961–981, 1982.

Peter C Fishburn. Non-transitive measurable utility for decision under uncertainty. *Journal of Mathematical Economics*, 18(2):187–207, 1989.

Botond Kőszegi and Adam Szeidl. A model of focusing in economic choice. *The Quarterly journal of economics*, 128(1):53–104, 2013.

Graham Loomes and Robert Sugden. Regret theory: An alternative theory of rational choice under uncertainty. *The economic journal*, 92(368):805–824, 1982.

Pedro Bordalo, Nicola Gennaioli, and Andrei Shleifer. Salience theory of choice under risk. *The Quarterly journal of economics*, 127(3):1243–1285, 2012.

Robert Sugden. An axiomatic foundation for regret theory. *Journal of Economic Theory*, 60(1):159–180, 1993.

John Quiggin. Regret theory with general choice sets. *Journal of Risk and Uncertainty*, 8:153–165, 1994.

Enrico Diecidue and Jeeva Somasundaram. Regret theory: A new foundation. *Journal of Economic Theory*, 172:88–119, 2017.

Andrew Ellis and Yusufcan Masatlioglu. Choice with endogenous categorization. *The Review of Economic Studies*, 89(1):240–278, 2022.

Fabian Herweg and Daniel Müller. A comparison of regret theory and salience theory for decisions under risk. *Journal of Economic Theory*, 193:105226, 2021.

Peter C Fishburn. Skew symmetric additive utility with finite states. *Mathematical Social Sciences*, 19(2):103–115, 1990b.

Efe A Ok et al. Utility representation of an incomplete preference relation. *Journal of Economic Theory*, 104(2):429–449, 2002.

# A Proofs

## A.1 Proof of Theorem 1

**Preliminaries.** Let $X$ be a finite set with $|X| = n$, and let $\Delta \subseteq \mathbb{R}^{n\times n}$ denote the set of all joint probability distributions over $X \times X$:

$$\Delta = \left\{ \pi \in \mathbb{R}^{n\times n} \mid \pi_{ij} \geq 0 \text{ for all } i, j = 1, 2, \ldots, n; \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_{ij} = 1 \right\}$$

I define a subset $\Pi \subseteq \Delta$ to represent the DM's preference set (i.e., $\pi \in \Pi \Leftrightarrow$ the row is weakly preferred). For any $\pi \in \Delta$, let $\bar{\pi}$ to denote the transpose of $\pi$. $\bar{\Pi}$ is the set of transposes of matrices in $\Pi$:

$$\bar{\Pi} = \{\pi \in \Delta : \bar{\pi} \in \Pi\}.$$

The set of incomplete preferences is defined as:

$$I = \{\pi \in \Delta : \pi \notin \Pi \text{ and } \pi \notin \bar{\Pi}\}.$$

The set $\Delta$ is a closed and convex subset of $\mathbb{R}^{n^2}$. By Axioms 2 and 4, the preference set $\Pi \subseteq \Delta$ is convex and closed, respectively. Since the transpose operation is linear and continuous, it follows that $\bar{\Pi}$, the set of transposes of elements in $\Pi$, is also a closed and convex subset of $\Delta$. In contrast, the set of incomparable options $I = \Delta \backslash (\Pi \cup \bar{\Pi})$ is not necessarily closed or convex and may consist of disconnected components within $\Delta$.

**Necessity of the axioms.** Considering every $\phi \in \Phi$ is skew–symmetric, i.e., $\phi(y, x) = -\phi(x, y)$ for all $(x, y) \in X \times X$,

$$\sum_{(x,y)\in X\times X} \pi(x, y)\phi(x, y) = -\sum_{(x,y)\in X\times X} \bar{\pi}(x, y)\phi(x, y).$$

Reflexivity is necessary since if $\pi = \bar{\pi}$:

$$\sum_{(x,y)\in X\times X} \pi(x, y)\phi(x, y) = -\sum_{(x,y)\in X\times X} \bar{\pi}(x, y)\phi(x, y) = 0.$$

Therefore, the multi–utility evaluation assigns the same value to $\pi$ and its transpose, and in particular, this value is equal to zero for all $\phi \in \Phi$. Consequently, both $\pi$ and $\bar{\pi}$ satisfy the inequalities of the representation and hence belong to the preference set, establishing reflexivity.

For strong independence, if $\pi, \pi' \in \Pi$ and $\alpha \in (0,1)$, then for all $\phi \in \Phi$,

$$\sum_{(x,y)\in X\times X} (\alpha\pi + (1-\alpha)\pi')(x,y)\phi(x,y) =$$

$$\alpha \sum_{(x,y)\in X\times X} \pi(x,y)\phi(x,y) + (1-\alpha) \sum_{(x,y)\in X\times X} \pi'\phi(x,y) \geq 0.$$

If $\pi' \in \hat{\Pi}$:

$$\alpha \sum_{(x,y)\in X\times X} \pi(x,y)\phi(x,y) + (1-\alpha) \sum_{(x,y)\in X\times X} \pi'(x,y)\phi(x,y) > 0.$$

For the axiom 3, let $\pi \in \Pi$, $\pi' \in I$, and $\alpha \in (0,1)$. By the definition of $I$, there exists a nonempty subset $\Phi' \subsetneq \Phi$ such that

$$\sum_{(x,y)\in X\times X} \pi'(x,y)\,\phi(x,y) \geq 0 \quad \text{for all } \phi \in \Phi',$$

$$\sum_{(x,y)\in X\times X} \pi'(x,y)\,\phi(x,y) < 0 \quad \text{for all } \phi \in \Phi \setminus \Phi'.$$

Consider the convex combination $\alpha\pi + (1-\alpha)\pi'$. For each $\phi \in \Phi'$, linearity implies

$$\sum_{(x,y)\in X\times X} (\alpha\pi + (1-\alpha)\pi')(x,y)\phi(x,y) \geq 0.$$

Suppose, to the contrary, that all of these inequalities hold with equality. Then

$$\sum_{(x,y)\in X\times X} \big(\alpha\pi + (1-\alpha)\pi'\big)(x,y)\,\phi(x,y) = 0, \qquad \forall\,\phi \in \Phi',$$

which is equivalent to

$$\sum_{(x,y)\in X\times X} \pi'(x,y)\,\phi(x,y) = 0, \qquad \forall\,\phi \in \Phi',$$

contradicting the assumption $\pi' \in I$. Hence, there exists at least one $\phi^\star \in \Phi'$ such that

$$\sum_{(x,y)\in X\times X} \big(\alpha\pi + (1-\alpha)\pi'\big)(x,y)\,\phi^\star(x,y) > 0,$$

which implies

$$\alpha\pi + (1-\alpha)\pi' \notin \bar{\Pi}.$$

For Continuity, let $\pi \in \Delta \setminus \Pi$ and denote

$$L_{\phi^\star}(\pi) := \sum_{(x,y)\in X\times X} \pi(x,y)\,\phi^\star(x,y) < 0.$$

$X$ is finite, so $\Delta$ is a finite-dimensional simplex and all sums are finite. For any arbitrary $\pi$ in $\Delta \setminus \Pi$, there is some $\phi^\star \in \Phi$ with $L_{\phi^\star}(\pi) < 0$. The map $L_{\phi^\star} : \Delta \to \mathbb{R}$ is linear, hence continuous. The preimage of the open set $(-\infty, 0)$ under a continuous map is open, so

$$U := L_{\phi^\star}^{-1}((-\infty, 0))$$

is an open neighborhood of $\pi$. By definition, every element of $U$ fails the inequality required for being in $\Pi$, so $U \subset \Delta \setminus \Pi$. Therefore, $\Delta \setminus \Pi$ is open and $\Pi$ is closed.

**Sufficiency of the axioms.** To prove the sufficiency of the axioms, the core of the argument establishes that, under the given axioms, the set of incomplete preferences $I$ cannot have more than two disconnected components. Once this structural property is established, the representation follows directly from the number of such components.

The proof unfolds in three stages. First, the behavior of line intervals connecting pairs of points in $I$ is classified according to their interaction with the sets $\Pi$ and $\bar{\Pi}$. This classification is subsequently extended to triangular configurations by examining the intervals formed between each pair of three points contained in $I$. Finally, it is shown that the existence of more than two disconnected components in $I$ would necessitate a triangular configuration that violates the axioms. This contradiction excludes such a possibility and thereby completes the argument.

To bridge the gap between the outline above and the detailed steps of the proof, the concept of a line interval is first defined, as it provides the basic building block for classifying geometric configurations within $I$.

For any $\pi, \pi' \in \Delta$, the line interval $\pi\pi'$ is defined as the set of all convex combinations of $\pi$ and $\pi'$:

$$\pi\pi' = \{\lambda\pi + (1-\lambda)\pi' | \lambda \in (0,1)\}$$

Given the definition of a line interval, the possible positions of the interval $\pi\pi'$ for $\pi, \pi' \in I$, are now categorized with respect to the subsets in $\Delta$. The following lemma formalizes this classification.

**Lemma 1.** *For any $\pi, \pi' \in I$, the line interval $\pi\pi'$ belongs to exactly one of the following categories:*

- *Type 1 (Fully Contained): $\pi\pi' \cap \Pi = \varnothing$ & $\pi\pi' \cap \bar{\Pi} = \varnothing$*
- *Type 2 (Only Crossing $\hat{\Pi}$): $\pi\pi' \cap \Pi \neq \varnothing$ & $\pi\pi' \cap \bar{\Pi} = \varnothing$*
- *Type 3 (Only Crossing $\bar{\hat{\Pi}}$): $\pi\pi' \cap \Pi = \varnothing$ & $\pi\pi' \cap \bar{\Pi} \neq \varnothing$*
- *Type 4 (Only Crossing $\Pi \cap \bar{\Pi}$): $\pi\pi' \cap (\Pi \cap \bar{\Pi}) \neq \varnothing$ & $\pi\pi' \cap (\hat{\Pi} \cup \bar{\hat{\Pi}}) = \varnothing$*

*Proof.* $\Delta$ is partitioned into four subsets $\hat{\Pi}, \bar{\hat{\Pi}}, \Pi \cap \bar{\Pi}, I \subseteq \Delta$. For any $\pi, \pi' \in I$, I may have $\lambda\pi + (1-\lambda)\pi' \in I$ for all $\lambda \in (0,1)$ and the entire interval $\pi\pi'$ lies within $I$:

$$\text{Type 1: } \pi\pi' \subset I \Leftrightarrow \pi\pi' \cap \Pi = \varnothing \ \& \ \pi\pi' \cap \bar{\Pi} = \varnothing.$$

Otherwise, there exists at least one $\lambda^* \in (0,1)$ such that $\lambda^*\pi + (1-\lambda^*)\pi' \notin I$, i.e., the interval partially intersects $\Delta \backslash I$.

$$\pi^* = \lambda^*\pi + (1-\lambda^*)\pi'$$

Since $\pi^* \notin I$, $\pi^*$ belongs to one of the subsets $\hat{\Pi}$, $\bar{\hat{\Pi}}$, or $\Pi \cap \bar{\Pi}$.

If $\pi^* \in \hat{\Pi}$, then by axiom 3, $\pi\pi^* \cap \bar{\Pi} = \varnothing$ and $\pi^*\pi' \cap \bar{\Pi} = \varnothing$. For all $\lambda \in (0, \lambda^*) \cup (\lambda^*, 1)$, the convex combination $\lambda\pi + (1-\lambda)\pi'$ cannot be in $\bar{\Pi}$:

$$\text{Type 2: } \pi\pi' \cap \Pi \neq \varnothing \ \& \ \pi\pi' \cap \bar{\Pi} = \varnothing.$$

Since $\Pi$ is convex and closed by axioms 2 and 4, respectively, I can further argue that there exist $\lambda_1, \lambda_2, 0 < \lambda_1 \leq \lambda_2 < 1$ such that:

$$\forall \lambda \in [\lambda_1, \lambda_2] \Rightarrow \lambda\pi + (1-\lambda)\pi' \in \hat{\Pi}$$

$$\forall \lambda \in (0, \lambda_1) \cup (\lambda_2, 1) \Rightarrow \lambda\pi + (1-\lambda)\pi' \in I$$

Likewise, if $\pi^*$ is in $\hat{\bar{\Pi}}$, then by axiom 3, $\pi\pi^* \cap \Pi = \varnothing$ and $\pi^*\pi' \cap \Pi = \varnothing$. For all $\lambda \in (0, \lambda^*) \cup (\lambda^*, 1)$, the convex combination $\lambda\pi + (1-\lambda)\pi'$ cannot be in $\Pi$:

$$\text{Type 3: } \pi\pi' \cap \Pi = \varnothing \ \& \ \pi\pi' \cap \bar{\Pi} \neq \varnothing.$$

Since $\bar{\Pi}$ is a set of all transposes of matrices in $\Pi$, $\bar{\Pi}$ is also convex and closed and I can further argue that there exist $\lambda_1, \lambda_2, 0 < \lambda_1 \leq \lambda_2 < 1$ such that:

$$\forall \lambda \in [\lambda_1, \lambda_2] \Rightarrow \lambda\pi + (1-\lambda)\pi' \in \hat{\bar{\Pi}}$$

$$\forall \lambda \in (0, \lambda_1) \cup (\lambda_2, 1) \Rightarrow \lambda\pi + (1-\lambda)\pi' \in I$$

The final case is when $\pi^* \in (\Pi \cap \bar{\Pi})$. By axiom 3, since $\pi^*$ lies in both $\Pi$ and $\bar{\Pi}$, the line intervals $\pi\pi^*$ and $\pi^*\pi'$ do not intersect either $\Pi$ or $\bar{\Pi}$, that is, $\pi\pi^* \cap (\Pi \cup \bar{\Pi}) = \varnothing$ and $\pi^*\pi' \cap (\Pi \cup \bar{\Pi}) = \varnothing$. It follows that for all $\lambda \in (0, \lambda^*) \cup (\lambda^*, 1)$, the convex combination $\lambda\pi + (1-\lambda)\pi'$ remains in $I$:

$$\text{Type 4: } \pi\pi' \cap (\Pi \cap \bar{\Pi}) \neq \varnothing \ \& \ \pi\pi' \cap (\hat{\Pi} \cup \hat{\bar{\Pi}}) = \varnothing$$

More precisely, in the final case, if there exists a point $\pi^* \in (\Pi \cap \bar{\Pi})$, then that point must be unique:

$$\lambda^*\pi + (1-\lambda^*)\pi' \in (\Pi \cap \bar{\Pi}), \lambda^* \in (0,1) \Rightarrow \forall \lambda \in (0, \lambda^*) \cup (\lambda^*, 1) : \lambda\pi + (1-\lambda)\pi' \in I$$

■

By Lemma 1, the line interval $\pi\pi'$, with $\pi, \pi' \in I$, cannot simultaneously intersect more than one of the subsets $\hat{\Pi}$, $\hat{\bar{\Pi}}$, and $\Pi \cap \bar{\Pi}$ within $\Delta$. Throughout the proof, the notion of *type* is used to classify line intervals connecting pairs of points within the incomplete subset $I$.

**Lemma 2.** *For all $\pi \in I$, the interval $\pi\bar{\pi}$ corresponds to type 4 in Lemma 1, as it contains a unique point in $\Pi \cap \bar{\Pi}$ and intersects both $\Pi$ and $\bar{\Pi}$ only at that point.*

*Proof.* $\frac{1}{2}\pi + \frac{1}{2}\bar{\pi}$ is equal to its transpose and by axiom 1, $\frac{1}{2}\pi + \frac{1}{2}\bar{\pi} \in (\Pi \cap \bar{\Pi})$. Therefore, $\lambda\pi + (1-\lambda)\bar{\pi}$ is in $\Pi \cap \bar{\Pi}$ for $\lambda = \frac{1}{2}$ which corresponds to type 4 from lemma 1:

$$\frac{1}{2}\pi + \frac{1}{2}\bar{\pi} \in (\Pi \cap \bar{\Pi})$$

$$\forall \lambda \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \Rightarrow \lambda \pi + (1-\lambda)\bar{\pi} \in I$$

■

**Lemma 3.** *For all $\pi, \pi', \pi'' \in I$ such that the intersection of $\Pi \cup \bar{\Pi}$ and each $\pi\pi', \pi\pi''$, and $\pi'\pi''$ is nonempty ($\pi\pi' \cap (\Pi \cup \bar{\Pi}) \neq \varnothing$, $\pi\pi'' \cap (\Pi \cup \bar{\Pi}) \neq \varnothing$, and $\pi'\pi'' \cap (\Pi \cup \bar{\Pi}) \neq \varnothing$), then either all $\pi\pi', \pi\pi''$, and $\pi'\pi''$ have intersection with $\hat{\Pi}$ ($\pi\pi' \cap \hat{\Pi} \neq \varnothing$, $\pi\pi'' \cap \hat{\Pi} \neq \varnothing$, and $\pi'\pi'' \cap \hat{\Pi} \neq \varnothing$) or all $\pi\pi', \pi\pi''$, and $\pi'\pi''$ have intersection with $\hat{\bar{\Pi}}$ ($\pi\pi' \cap \hat{\bar{\Pi}} \neq \varnothing$, $\pi\pi'' \cap \hat{\bar{\Pi}} \neq \varnothing$, and $\pi'\pi'' \cap \hat{\bar{\Pi}} \neq \varnothing$). In other words, according to lemma 1, all $\pi\pi', \pi\pi''$, and $\pi'\pi''$ have the same type and it's either 2 or 3.*

*Proof.* $\pi, \pi', \pi'' \in I$, and the intersection of $\Pi \cup \bar{\Pi}$ and each line interval $\pi\pi', \pi\pi''$, and $\pi'\pi''$ is nonempty; therefore, $\pi\pi', \pi\pi''$, and $\pi'\pi''$ can have one of the types 2, 3, or 4 from lemma 1. Having one of the three types for each side of the triangle, there are ten possibilities for the type profile of the triangle $\pi\pi'\pi''$, one when no two sides of the triangle have the same type, six when exactly two sides have the same type, and three when all sides have the same type.

If all sides of the triangle have different types, without loss of generality I can assume $\pi\pi' \cap (\Pi \cap \bar{\Pi}) \neq \varnothing$, $\pi\pi'' \cap \hat{\Pi} \neq \varnothing$, and $\pi'\pi'' \cap \hat{\bar{\Pi}} \neq \varnothing$.

$$\exists \lambda_1 \in (0,1) : \pi_1 = \lambda_1 \pi + (1-\lambda_1)\pi' \in \Pi \cap \bar{\Pi}$$

$$\exists \lambda_2 \in (0,1) : \pi_2 = \lambda_2 \pi + (1-\lambda_2)\pi'' \in \hat{\Pi}$$

$$\exists \lambda_3 \in (0,1) : \pi_3 = \lambda_3 \pi' + (1-\lambda_3)\pi'' \in \hat{\bar{\Pi}}$$

By axiom 2, $\pi_1\pi_2 \subset \hat{\Pi}$ and by axiom 3, $\pi\pi_3 \cap \Pi = \varnothing$, however, this is a contradiction because $\pi_1\pi_2$ and $\pi\pi_3$ have an intersection, $\pi_1\pi_2 \cap \pi\pi_3 \neq \varnothing$. Hence, it is impossible to construct a triangle $\pi\pi'\pi''$ in which each side is of a distinct type as classified in Lemma 1, see figure 2.

If exactly two sides have type 4, I can assume $\pi\pi' \cap (\Pi \cap \bar{\Pi}) \neq \varnothing$, $\pi\pi'' \cap (\Pi \cap \bar{\Pi}) \neq \varnothing$, and $\pi'\pi'' \cap \hat{\Pi} \neq \varnothing$.

$$\exists \lambda_1 \in (0,1) : \pi_1 = \lambda_1 \pi + (1-\lambda_1)\pi' \in \Pi \cap \bar{\Pi}$$

$$\exists \lambda_2 \in (0,1) : \pi_2 = \lambda_2 \pi + (1-\lambda_2)\pi'' \in \Pi \cap \bar{\Pi}$$

$$\exists \lambda_3 \in (0,1) : \pi_3 = \lambda_3 \pi' + (1-\lambda_3)\pi'' \in \hat{\Pi}$$

By axiom 2, $\pi_1\pi_2 \subset \Pi \cap \bar{\Pi}$ and by axiom 3, $\pi\pi_3 \cap \bar{\Pi} = \varnothing$, however, this is

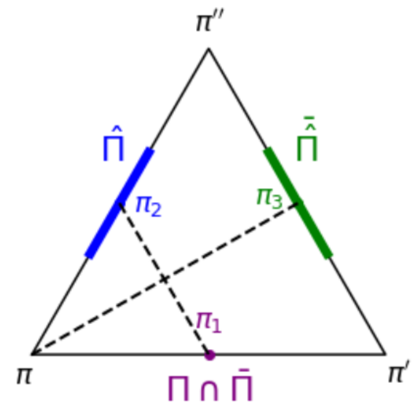


Figure 2: Triangle $\pi\pi'\pi''$ with a type profile where each side, $\pi\pi'$, $\pi\pi''$, and $\pi'\pi''$, belongs to a different type as defined in Lemma 1. A contradiction arises because the intersection of $\pi_1\pi_2$ and $\pi\pi_3$ is required to belong to $\hat{\Pi}$, while it must not lie in $\Pi$. The same reasoning applies when $\pi\pi'$ is of type 2 and $\pi_1 \in \hat{\Pi}$.

a contradiction because $\pi_1\pi_2$ and $\pi\pi_3$ have an intersection, $\pi_1\pi_2 \cap \pi\pi_3 \neq \varnothing$. Hence, it is impossible to construct a triangle $\pi\pi'\pi''$ in which exactly two sides have type 4 as classified in Lemma 1, see figure 3. If $\pi'\pi'' \cap \bar{\hat{\Pi}} \neq \varnothing$ the same reasoning applies.

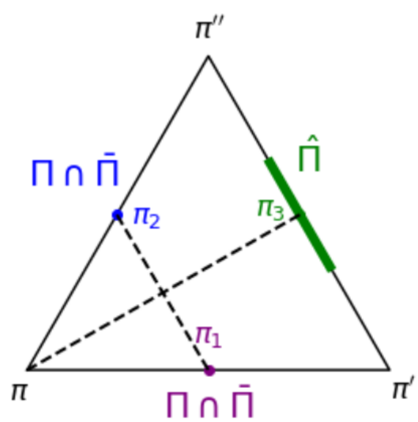


Figure 3: Triangle $\pi\pi'\pi''$ with a type profile in which two sides share the same type (as defined in Lemma 1). A contradiction occurs because the intersection of $\pi_1\pi_2$ and $\pi\pi_3$ is required to belong to $\Pi \cap \bar{\Pi}$, yet it cannot lie in $\bar{\Pi}$.

If exactly two sides have type 2, I can assume $\pi\pi' \cap \hat{\Pi} \neq \varnothing$, $\pi\pi'' \cap \hat{\Pi} \neq \varnothing$, and $\pi'\pi'' \cap \bar{\hat{\Pi}} \neq \varnothing$.

$$\exists \lambda_1 \in (0,1) : \pi_1 = \lambda_1 \pi + (1-\lambda_1)\pi' \in \hat{\Pi}$$

$$\exists \lambda_2 \in (0,1) : \pi_2 = \lambda_2 \pi + (1-\lambda_2)\pi'' \in \hat{\Pi}$$

$$\exists \lambda_3 \in (0,1) : \pi_3 = \lambda_3 \pi' + (1-\lambda_3)\pi'' \in \bar{\hat{\Pi}}$$

By axiom 2, $\pi_1\pi_2 \subset \hat{\Pi}$ and by axiom 3, $\pi\pi_3 \cap \Pi = \varnothing$, however, this is a contradiction because $\pi_1\pi_2$ and $\pi\pi_3$ have an intersection, $\pi_1\pi_2 \cap \pi\pi_3 \neq \varnothing$. If $\pi'\pi'' \cap (\Pi \cap \bar{\Pi}) \neq \varnothing$ the same reasoning applies.

If exactly two sides have type 3, this is similar to the previous example, and I can assume $\pi\pi' \cap \bar{\hat{\Pi}} \neq \varnothing$, $\pi\pi'' \cap \bar{\hat{\Pi}} \neq \varnothing$, and $\pi'\pi'' \cap \hat{\Pi} \neq \varnothing$.

$$\exists \lambda_1 \in (0,1) : \pi_1 = \lambda_1 \pi + (1-\lambda_1)\pi' \in \bar{\hat{\Pi}}$$

$$\exists \lambda_2 \in (0,1) : \pi_2 = \lambda_2 \pi + (1-\lambda_2)\pi'' \in \bar{\hat{\Pi}}$$

$$\exists \lambda_3 \in (0,1) : \pi_3 = \lambda_3 \pi' + (1-\lambda_3)\pi'' \in \hat{\Pi}$$

By axiom 2, $\pi_1\pi_2 \subset \bar{\hat{\Pi}}$ and by axiom 3, $\pi\pi_3 \cap \bar{\Pi} = \varnothing$, however, this is a contradiction because $\pi_1\pi_2$ and $\pi\pi_3$ have an intersection, $\pi_1\pi_2 \cap \pi\pi_3 \neq \varnothing$. If $\pi'\pi'' \cap (\Pi \cap \bar{\Pi}) \neq \varnothing$ the same reasoning applies.

If all sides of the triangle have type 4, I can assume $\pi\pi' \cap (\Pi \cap \bar{\Pi}) \neq \varnothing$, $\pi\pi'' \cap (\Pi \cap \bar{\Pi}) \neq \varnothing$, and $\pi'\pi'' \cap (\Pi \cap \bar{\Pi}) \neq \varnothing$.

$$\exists \lambda_1 \in (0,1) : \pi_1 = \lambda_1 \pi + (1-\lambda_1)\pi' \in \Pi \cap \bar{\Pi}$$

$$\exists \lambda_2 \in (0,1) : \pi_2 = \lambda_2 \pi + (1-\lambda_2)\pi'' \in \Pi \cap \bar{\Pi}$$

$$\exists \lambda_3 \in (0,1) : \pi_3 = \lambda_3 \pi' + (1-\lambda_3)\pi'' \in \Pi \cap \bar{\Pi}$$

By axiom 2, $\pi_1\pi_2 \subset (\Pi \cap \bar{\Pi})$ and by axiom 3, $\pi\pi_3 \cap (\Pi \cup \bar{\Pi}) = \varnothing$, however, this is a contradiction because $\pi_1\pi_2$ and $\pi\pi_3$ have an intersection, $\pi_1\pi_2 \cap \pi\pi_3 \neq \varnothing$. So, I cannot have a triangle $\pi\pi'\pi''$ such that each side has type 4 from lemma 1.

Suppose all sides of the triangle are of the same type, either type 2 or type 3. In this case, no contradiction with the axioms arises, since the interior of the triangle contains a convex region that belongs entirely to either $\hat{\Pi}$ or $\bar{\hat{\Pi}}$. Moreover, by Axiom 3, the surrounding region may lie in the incomplete subspace without violating any of the stated conditions. ∎

Lemma 3 states that if there exist three distinct points in $I$ such that the convex combination of each pair is not entirely contained in the incomplete subset (i.e., none of the corresponding line intervals has type 1 as defined in Lemma1), then all three line intervals must share the same type. Moreover, this common type can only be type 2 or type 3 according to the classification in Lemma 1.

The next step is to show that $I$ contains at most two distinct, disjoint components. To formalize this, it is necessary to define the components of $I$.

**Same-component relation $R$ on $I$:** For any $\pi_1, \pi_n \in I$, define $\pi_1 R \pi_n$ if there exists a finite sequence $\{\pi_1, \pi_2, \ldots, \pi_n\} \subset I$ such that

$$\forall \alpha \in (0,1),\ \forall i \in \{1, \ldots, n-1\} : \quad \alpha\pi_i + (1-\alpha)\pi_{i+1} \in I.$$

That is, $\pi_1$ and $\pi_n$ are in the same path-connected component of $I$ with respect to convex combinations (piecewise-linear paths entirely contained in $I$). The path is realized by piecewise linear segments connecting $\pi_1$ to $\pi_n$ entirely within $I$.

**Component of $I$:** A subset $C \subset I$ is a *component* if it is maximal with respect to the same-component relation $R$, that is,

$\forall \pi_1, \pi_2 \in C : \pi_1 R \pi_2$, and there exists no $\pi \in I \setminus C$ such that $\pi R \pi_1$ for some $\pi_1 \in C$.

By definition, components are disjoint, path-connected subsets of $I$ under convex-combination paths, and every point in $I$ belongs to exactly one component.

**Lemma 4.** *There are no more than two components in $I$. For any $\pi, \pi', \pi'' \in I$, if $\pi \not R \pi'$ and $\pi \not R \pi''$, then $\pi' R \pi''$.*

*Proof.* The lemma holds if there is none or only one component in $I$. Otherwise, assume there exist at least two disjoint incomplete components. Let $\pi, \pi' \in I$ belong to different components, so that $\pi \not R \pi'$. By Lemma 1, the line interval between $\pi$ and $\pi'$, as well as the one between $\bar{\pi}$ and $\bar{\pi}'$, must be of one of Types 2, 3, or 4. Furthermore, by Lemma 2, the interval $\pi\bar{\pi}$ is of Type 4 in the sense of Lemma 1. By lemma 3, it is not possible to form a triangle $\pi\bar{\pi}\pi'$ such that

$\bar{\pi}\pi' \not\subset I$, given that $\pi\bar{\pi}$ is of type 4. Moreover, observe that $\pi'\bar{\pi}$ being entirely in $I$ is determined by that of $\pi\bar{\pi}'$: if one lies entirely in $I$, so does the other. Indeed, the line interval $\pi\bar{\pi}', \alpha \in (0,1) : \alpha\pi + (1-\alpha)\bar{\pi}'$ consists of the transposes of the line interval $\pi'\bar{\pi}, \beta \in (0,1) : \beta\bar{\pi} + (1-\beta)\pi'$.

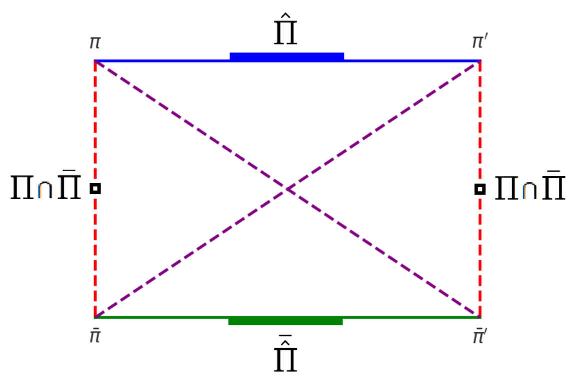


Figure 4: Without loss of generality, the interval between $\pi$ and $\pi'$ is assumed to be of type 2, as described in Lemma 1. This implies that the interval between $\bar{\pi}$ and $\bar{\pi}'$ is of type 3. The intervals $\pi\bar{\pi}'$ and $\pi'\bar{\pi}$ both lie in the incomplete subset.

Now assume there is another component in $I$, and $\pi''$ belongs to that component. Specifically, $\pi''$ is neither in the component of $\pi$ nor in that of $\pi'$ ($\pi'' \not R\, \pi$, $\pi'' \not R\, \pi'$). Since $\pi'' \not R\, \pi$, the line $\pi''\pi$ can be any of the types described in Lemma 1, except type 1. As $\bar{\pi}$ is in the same component as $\pi'$, it follows that $\pi'' \not R\, \bar{\pi}$, and the line $\pi''\bar{\pi}$ is one of the types in Lemma 1, except type 1. Now consider the triangle $\pi\bar{\pi}\pi''$. By Lemma 3, such a triangle cannot exist because $\pi\bar{\pi}$ is of type 4 according to Lemma 1. This contradiction implies that there cannot be more than two components in $I$. ■

**Convexity of all components in $I$:** For any $\pi, \pi' \in I$ and $\alpha \in (0,1)$, if $\pi$ and $\pi'$ belong to the same component, $\pi R \pi'$, then

$$\alpha\pi + (1-\alpha)\pi' \in I.$$

This property ensures that each component of $I$ is convex: any convex combination of two points in the same component lies in $I$ and remains in that component.

**Lemma 5.** *If there are exactly two nonempty components in $I$, each component* *is convex and for all $\pi \in I$: $\pi \not R \bar{\pi}$.*

*Proof.* Let $I$ contain exactly two components, and let $\pi, \pi' \in I$ belong to different components, so that $\pi \not R \pi'$. By Lemma 3, the intervals $\pi'\bar{\pi}$ and $\pi\bar{\pi}'$ lie in the incomplete subset, with $\pi' R\bar{\pi}$ and $\pi R\bar{\pi}'$. Consider a point $\pi'' \in I$ such that $\pi'' R\pi'$ and $\pi'' \not R \pi$. In this case, the interval $\pi''\pi$ must be one of the types described in Lemma 1, excluding type 1. Since $\pi'' R\pi'$ permits any type, assume initially that $\pi''\pi'$ is of a type other than type 1. Considering the triangle $\pi'\bar{\pi}'\pi''$, with $\bar{\pi}\bar{\pi}'$ of type 4 and $\pi''\pi'$ assumed to have any type other than type 1, Lemma 3 requires that $\bar{\pi}'\pi'' \subset I$, which implies $\bar{\pi}' R\pi''$. This, however, contradicts the assumption $\pi'' \not R \pi$ since $\pi\bar{\pi}' \subset I$. Therefore, the interval $\pi''\pi'$ must be of type 1 if $\pi'' R\pi'$, so that $\pi''\pi' \subset I$. Consequently, $\pi'' R\pi'$ coincides with $\pi''\pi' \subset I$, and the transpose of the matrix does not belong to the same component as the matrix itself. ■

In general, $\Delta$ may contain zero, one, or two disjoint incomplete components. In the case of zero incomplete components, the preference is complete. The sets $\Pi$ and $\bar{\Pi}$ then cover the entire $\Delta$, and both are closed and convex. This configuration ensures the existence of a separating hyperplane, $\phi$, separating $\Pi$ and $\bar{\Pi}$:

$$\pi \in \Pi \Leftrightarrow \sum_{(x,y)\in X\times X} \pi(x,y)\phi(x,y) \geq 0.$$

In the case of two incomplete components, the convexity of each component, together with the convexity of the union of the both the preference set and $\bar{\Pi}$ with each incomplete component, implies the existence of two hyperplanes.

$$\pi \in \Pi \Leftrightarrow \begin{cases} \sum_{(x,y)\in X\times X} \pi(x,y)\phi_1(x,y) \geq 0; \\ \& \\ \sum_{(x,y)\in X\times X} \pi(x,y)\phi_2(x,y) \geq 0. \end{cases}$$

Moreover, $\Pi$ and $\bar{\Pi}$ lie on opposite sides of each hyperplane. There are four primary convex regions: $\Pi$, $\bar{\Pi}$, and two disjoint components of $I$, denoted $I_1$ and $I_2$. The unions $\Pi \cup I_1$ and $\bar{\Pi} \cup I_2$ are convex and together cover the entire $\Delta$, as do the unions $\Pi \cup I_2$ and $\bar{\Pi} \cup I_1$.

So I need $\phi_1$ and $\phi_2$ such that:

$$\pi \in \Pi \Leftrightarrow \begin{cases} \sum_{(x,y)\in X\times X} \bar{\pi}(x,y)\phi_1(x,y) \leq 0; \\ \& \\ \sum_{(x,y)\in X\times X} \bar{\pi}(x,y)\phi_2(x,y) \leq 0. \end{cases}$$

$\Delta$ is partitioned into these 4 subsets:

1. $\pi \in \Pi$: $\sum_{(x,y)\in X\times X} \pi(x,y)\phi_1(x,y) \geq 0$, $\sum_{(x,y)\in X\times X} \pi(x,y)\phi_2(x,y) \geq 0$;
2. $\pi \in \bar{\Pi}$: $\sum_{(x,y)\in X\times X} \pi(x,y)\phi_1(x,y) \leq 0$, $\sum_{(x,y)\in X\times X} \pi(x,y)\phi_2(x,y) \leq 0$;
3. $\pi \in I_1$: $\sum_{(x,y)\in X\times X} \pi(x,y)\phi_1(x,y) \geq 0$, $\sum_{(x,y)\in X\times X} \pi(x,y)\phi_2(x,y) \leq 0$;
4. $\pi \in I_2$: $\sum_{(x,y)\in X\times X} \pi(x,y)\phi_1(x,y) \leq 0$, $\sum_{(x,y)\in X\times X} \pi(x,y)\phi_2(x,y) \geq 0$.

For any $\pi \in \Delta$, if $\pi$ belongs to $\Pi$, then $\bar{\pi}$ belongs to $\bar{\Pi}$; if $\pi$ belongs to $I_1$, then $\bar{\pi}$ belongs to $I_2$, and vice versa. From the defining inequalities, it follows that for each $\pi \in \Delta$, the transpose $\bar{\pi}$ lies on the opposite side of both $\phi_1$ and $\phi_2$.

Define the property for $\phi : X \times X \to \mathbb{R}$ as follows:

$$\forall \pi \in \Delta : \sum_{(x,y)\in X\times X} \pi(x,y)\phi(x,y) \geq 0 \Rightarrow \sum_{(x,y)\in X\times X} \bar{\pi}(x,y)\phi(x,y) \leq 0.$$

**Lemma 6.** *$\phi : X \times X \to \mathbb{R}$ is skew-symmetric, $\bar{\phi} = -\phi$, iff for any $\pi \in \Delta$*

$$\sum_{(x,y)\in X\times X} \pi(x,y)\phi(x,y) \geq 0 \Rightarrow \sum_{(x,y)\in X\times X} \bar{\pi}(x,y)\phi(x,y) \leq 0.$$

*Proof.* The sufficiency of the condition $\bar{\phi} = -\phi$ follows directly. To establish necessity, assume instead that $\bar{\phi} \neq -\phi$. Then, there exists at least one element with $\phi_{ij} \neq -\phi_{ji}$. Since the property must hold for all $\pi \in \Delta$, consider the case in which only $\pi_{ij}$ and $\pi_{ji}$ are nonzero. If $\phi_{ij} \neq -\phi_{ji}$, it is always possible to select $\pi_{ij}$ and $\pi_{ji}$ such that $\phi_{ij}\pi_{ij} + \phi_{ji}\pi_{ji}$ and $\phi_{ji}\pi_{ij} + \phi_{ij}\pi_{ji}$ share the same sign, which contradicts the initial requirement. ■

Therefore, to ensure that a hyperplane $\phi$ satisfies the requirement that $\pi$ and $\bar{\pi}$ lie on opposite sides of $\phi$ for every $\pi \in \Delta$, condition $\bar{\phi} = -\phi$ must hold. $C(X \times X)$ is the space of all real-valued functions on $X \times X$ and the subspace of skew-symmetric functions is represented as

$$C^{ss}(X \times X) = \{c \in C(X \times X) : c(x, y) = -c(y, x) \text{ for all } x, y \in X\}.$$

Having two components in $I$ is equivalent to two hyperplanes ,$\phi_1$ and $\phi_2$, each satisfying $\bar{\phi_1} = -\phi_1$ and $\bar{\phi_2} = -\phi_2$ or in other words $\phi_1, \phi_2 \in C^{ss}$. This can be interpreted as the presence of two distinct evaluation systems: each hyperplane represents one system's assessment of the pairs in $\Delta$. Preference is incomplete in this setting when the two systems produce contradictory evaluations.

For a single-component $I$, the space $\Delta$ consists of three subsets: $\Pi$, $\bar{\Pi}$, and $I$, with $\Pi$ and $\bar{\Pi}$ overlapping. Both $\Pi$ and $\bar{\Pi}$ are closed and convex, and by Axiom 1, their intersection $\Pi \cap \bar{\Pi}$ is nonempty. Furthermore, Axiom 3 guarantees that for each $\pi \in I$, a non-strict separating hyperplane $\phi$ exists that separates $\Pi$ and $\bar{\Pi}$, as established in the next lemma.

**Lemma 7.** *For any $\pi \in I$, there exists a non-strict separating hyperplane $\phi$, $\phi : X \times X \to \mathbb{R}$, that separates $\Pi$ and $\bar{\Pi}$ such that for all of them, every $\tilde{\pi} \in (\pi \cup (\Pi \cap \bar{\Pi}))$ lies on $\phi$:*

$$\sum_{(x,y) \in X \times X} \tilde{\pi}(x, y)\phi(x, y) = 0.$$

*Proof.* $\Pi$ and $\bar{\Pi}$ are both closed and convex, and by Axiom 2 they overlap only at their boundaries, meaning that their interiors, denoted $\text{int}(\Pi)$ and $\text{int}(\bar{\Pi})$, are disjoint. Therefore, the Minkowski Separation Theorem ensures the existence of a hyperplane that separates $\Pi$ and $\bar{\Pi}$; however, this does not guarantee that for every $\pi \in I$, there exists a separating hyperplane that also passes through $\pi$.

Let $\pi \in I$ and $\pi' \in \Pi$. By Axiom 3, for any $\alpha \in (0, 1)$, $\alpha\pi + (1 - \alpha)\pi' \notin \bar{\Pi}$. The set $\text{conv}(\pi, \hat{\Pi})$ denotes the convex hull of $\pi$ and all points in $\hat{\Pi}$ and by axiom 3, since $\pi$ is in $I$, $\text{conv}(\pi, \hat{\Pi}) \cup \bar{\Pi} = \varnothing$. The sets $\text{conv}(\pi, \hat{\Pi})$ and $\text{conv}(\pi, \hat{\bar{\Pi}})$ overlap only at $\pi$, and by the Separation Theorem, there exists a non-strict separating $\phi$ between them that contains $\pi$. $\hat{\Pi} \subset \text{conv}(\pi, \hat{\Pi})$ and $\hat{\bar{\Pi}} \subset \text{conv}(\pi, \hat{\bar{\Pi}})$. Consequently, the separating hyperplane that contains $\pi$ also separates $\hat{\Pi}$ and $\hat{\bar{\Pi}}$ (non-strictly),

$$L_\phi(\pi) = \sum_{(x,y) \in X \times X} \pi(x, y)\phi(x, y) = 0.$$

Every point of $\Delta$ lies either on this hyperplane or on one of its two sides. Without loss of generality, assume that for every $\pi' \in \hat{\Pi}$

$$L_\phi(\pi') = \sum_{(x,y)\in X\times X} \pi'(x,y)\phi(x,y) \geq 0,$$

so that every $\pi'' \in \bar{\hat{\Pi}}$ satisfies $L_\phi(\pi'') \geq 0$.

Let $\pi^\star \in (\Pi \cap \bar{\Pi})$ and suppose, for contradiction that $L_\phi(\pi^\star) < 0$. Choose $\pi' \in \hat{\Pi}$ so that $L_\phi(\pi') \geq 0$. By Axiom 2 every convex combination of $\pi^\star$ with a point of $\hat{\Pi}$ lies in $\hat{\Pi}$, it follows that

$$L_\phi(\alpha\pi^\star + (1-\alpha)\pi') \geq 0 \qquad \forall\alpha \in (0,1).$$

Linearity of $L_\phi$ yields

$$L_\phi(\alpha\pi^\star + (1-\alpha)\pi') = \alpha L_\phi(\pi^\star) + (1-\alpha)L_\phi(\pi'),$$

hence the inequality above becomes

$$\alpha L_\phi(\pi^\star) + (1-\alpha)L_\phi(\pi') \geq 0 \qquad \forall\alpha \in (0,1).$$

Taking the limit as $\alpha \to 1$ (or, equivalently, observing that the left–hand side is affine in $\alpha$) gives $L_\phi(\pi^\star) \geq 0$, contradicting the assumption $L_\phi(\pi^\star) < 0$. Therefore $L_\phi(\pi^\star) = 0$. If instead $L_\phi(\pi^\star) > 0$, choose $\pi'' \in \bar{\hat{\Pi}}$ so that $L_\phi(\pi'') \leq 0$; linearity of $L_\phi$, together with axiom 2, which states that every convex combination of $\pi^\star$ and a point of $\bar{\hat{\Pi}}$ belongs to $\bar{\hat{\Pi}}$, forces a contradiction unless $L_\phi(\pi^\star) = 0$. So every $\pi^\star \in \Pi \cap \bar{\Pi}$ lies on the separating hyperplane. ■

By Lemma 7, for any hyperplane separating $\Pi$ and $\bar{\Pi}$, if $\pi \in \Pi$ (or $\pi \in \bar{\Pi}$), its counterpart $\bar{\pi}$ lies on the opposite side of the hyperplane. Moreover, if $\pi \in I$, then $\frac{1}{2}\pi + \frac{1}{2}\bar{\pi} \in (\Pi \cap \bar{\Pi})$ lies on the hyperplane, and by the convexity of the partition induced by the hyperplane in $\Delta$, it follows that $\bar{\pi}$ is on the opposite side. Since each $\pi \in \Delta$ has its counterpart $\bar{\pi}$ located on the opposite side, Lemma 6 implies that these separating hyperplanes must be skew-symmetric, i.e., $\phi \in C^{ss}$.

The preference set $\Pi$ is convex and closed, and by the supporting–hyperplane theorem, every boundary point $p \in \partial\Pi$ admits a supporting hyperplane $\phi$,

$$L_\phi(p) = \sum_{(x,y)\in X\times X} p(x,y)\phi(x,y) = 0.$$

Ignoring the hyperplanes on the boundary of $\Delta$, since they reveal no information on the structure of $\Pi$, $\Phi$ is the set of all such hyperplanes and $p \in \text{int}(\Delta) \cap \partial\Pi$.

**Lemma 8.** *Every $\phi \in \Phi$ defines a non-strict separating hyperplane that separates $\Pi$ and $\bar{\Pi}$, and it is skew-symmetric. In particular, $\bar{\phi} = -\phi$, so $\phi \in C^{ss}$.*

*Proof.* Let $p \in \text{int}(\Delta) \cap \partial\Pi$ and $\phi$ be the supporting hyperplane, $L_\phi(p) = 0$ and for all $\pi \in \Pi$, $L_\phi(\pi) \geq 0$. Suppose, for contradiction, that there exists $p' \in \bar{\hat{\Pi}}$ such that $L_\phi(p') > 0$. Since $p \in \text{int}(\Delta)$, for $\alpha \in (0,1)$, $\alpha \to 1$:

$$p'' = \frac{p - (1-\alpha)p'}{\alpha} \to p,$$

$$L_\phi(p'') = \frac{1}{\alpha}L_\phi(p) - \frac{1-\alpha}{\alpha}L_\phi(p') \to 0^-.$$

$L_\phi(p'') < 0$ so $p'' \notin \Pi$. Since $\alpha p'' + (1-\alpha)p' = p$, if $p'' \in I$, by axiom 3, $p \notin \Pi$, otherwise if $p'' \in \bar{\hat{\Pi}}$, by axiom 2, $p \notin \Pi$ and both cases contradicts with the initial assumption that $p \in \Pi$. Therefore, there is no $p' \in \bar{\hat{\Pi}}$ such that $L_\phi(p') > 0$. For any $\pi \in \Pi \cap \bar{\Pi}$, $L_\phi(\pi) = 0$ and for any $\pi \in I$, since $\frac{1}{2}\pi + \frac{1}{2}\bar{\pi} \in (\Pi \cap \bar{\Pi})$ lies on the hyperplane, $\pi$ and $\bar{\pi}$ are not in the same side of the hyperplane. Thus, by lemma 6, $\phi \in C^{ss}$. ■

$\Phi$ is the collection of the supporting hyperplanes that characterize $\Pi$ within $\Delta$ and $\Phi \subset C^{ss}$:

$$\pi \in \Pi \Leftrightarrow \sum_{(x,y)\in X\times X} \pi(x,y)\phi(x,y) \geq 0 \quad \text{for all } \phi \in \Phi.$$

## A.2 Proof of Theorem 2

Let $\Pi$ admit an incomplete correlation-sensitive representation; that is, there exists a nonempty subset $\Phi \subset C^{ss}(X \times X)$ that represents $\Pi$. Consider a joint distribution $\pi_{ABC} \in \Delta(X^3)$ over three options, and denote its pairwise marginals by $\pi_{AB}$, $\pi_{BC}$, and $\pi_{AC}$.

$$\pi_{AB} \in \Pi \Leftrightarrow \sum_{(x,y)\in X\times X} \pi_{AB}(x,y)\phi(x,y) \geq 0 \quad \forall\phi \in \Phi,$$

$$\pi_{BC} \in \Pi \Leftrightarrow \sum_{(x,y)\in X\times X} \pi_{BC}(x,y)\phi(x,y) \geq 0 \quad \forall \phi \in \Phi,$$

$$\pi_{AC} \in \Pi \Leftrightarrow \sum_{(x,y)\in X\times X} \pi_{AC}(x,y)\phi(x,y) \geq 0 \quad \forall \phi \in \Phi.$$

The pairwise marginals can be determined in terms of $\pi_{ABC}$, allowing the inequalities to be rewritten as

$$\pi_{AB} \in \Pi \Leftrightarrow \sum_{(x,y,z)\in X^3} \pi_{ABC}(x,y,z)\phi(x,y) \geq 0 \quad \forall \phi \in \Phi,$$

$$\pi_{BC} \in \Pi \Leftrightarrow \sum_{(x,y,z)\in X^3} \pi_{ABC}(x,y,z)\phi(y,z) \geq 0 \quad \forall \phi \in \Phi,$$

$$\pi_{AC} \in \Pi \Leftrightarrow \sum_{(x,y,z)\in X^3} \pi_{ABC}(x,y,z)\phi(x,z) \geq 0 \quad \forall \phi \in \Phi.$$

Strong transitivity requires that if $\pi_{AB} \in \Pi$ and $\pi_{BC} \in \Pi$, then $\pi_{AC} \in \Pi$. To establish sufficiency, suppose that $\phi(x,z) = \phi(x,y)+\phi(y,z)$ for all $\phi \in \Phi$ and $x,y,z \in X$, where $L_\phi(\pi) = \sum_{(x,y)\in X\times X} \pi(x,y)\phi(x,y)$. Under this condition,

$$L_\phi(\pi_{AC}) = L_\phi(\pi_{AB}) + L_\phi(\pi_{BC}) \quad \forall \phi \in \Phi.$$

Hence if,

$$L_\phi(\pi_{AB}) \geq 0, \quad L_\phi(\pi_{BC}) \geq 0 \quad \forall \phi \in \Phi,$$

then

$$L_\phi(\pi_{AC}) \geq 0 \quad \forall \phi \in \Phi,$$

and therefore

$$\pi_{AB} \in \Pi, \quad \pi_{BC} \in \Pi \;\Rightarrow\; \pi_{AC} \in \Pi.$$

Moreover, since every $\phi \in \Phi$ is skew-symmetric, the condition

$$\phi(x,z) = \phi(x,y) + \phi(y,z)$$

holds automatically for all $x,y,z \in X$, whenever $X$ contains only two elements.

For necessity, it must be shown that a violation of strong transitivity requires at least three distinct outcomes in $X$ and the existence of some $\phi \in \Phi$ such that

$$\phi(x,z) \neq \phi(x,y) + \phi(y,z)$$

for some $x, y, z \in X$.

Strong transitivity requires that if for all $\phi \in \Phi$, $L_\phi(\pi_{AB}) \geq 0$ and $L_\phi(\pi_{BC}) \geq 0$, then for all $\phi \in \Phi$, $L_\phi(\pi_{AC}) \geq 0$. Consequently, if for all $\phi \in \Phi$, $L_\phi(\pi_{AB}) \geq 0$ and $L_\phi(\pi_{BC}) \geq 0$, but there exists some $\phi \in \Phi$ such that $L_\phi(\pi_{AC}) < 0$, then strong transitivity is violated. The remainder of the proof establishes the existence of such a case when $X$ contains at least three outcomes, $X = \{x, y, z\}$, and there exists $\phi \in \Phi$ such that $\phi(x, z) \neq \phi(x, y) + \phi(y, z)$.

Let $X = \{x, y, z\}$, and consider three options $A$, $B$, and $C$. In general, there are $3^3$ possible states corresponding to the combinations of outcomes for the three options. Denote these states by $s_i$, $i = 1, 2, \ldots, 27$. Given this state space and the associated probabilities $p_j$ for each state, the correlation structure among the options is fully characterized.

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $\ldots$ | $s_{10}$ | $s_{11}$ | $s_{12}$ | $\ldots$ | $s_{25}$ | $s_{26}$ | $s_{27}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $\ldots$ | $y$ | $y$ | $y$ | $\ldots$ | $z$ | $z$ | $z$ |
| $B$ | $x$ | $x$ | $x$ | $y$ | $y$ | $y$ | $\ldots$ | $x$ | $x$ | $x$ | $\ldots$ | $z$ | $z$ | $z$ |
| $C$ | $x$ | $y$ | $z$ | $x$ | $y$ | $z$ | $\ldots$ | $x$ | $y$ | $z$ | $\ldots$ | $x$ | $y$ | $z$ |

Table 2: Possible states for three options, having $X = \{x, y, z\}$

$\pi_{AB}$ denotes the correlation structure between options $A$ and $B$, with the outcomes of $A$ indexed by rows and those of $B$ indexed by columns. By the incomplete correlation-sensitive representation, the inequalities can be rewritten, and if $\pi_{AB}, \pi_{BC}, \pi_{CA} \in \Pi$, then there is a cycle unless $\pi_{AB}, \pi_{BC}, \pi_{CA} \in \Pi \cap \bar{\Pi}$.

$$\pi_{AB} \in \Pi \;\Leftrightarrow\; \sum_j p_j \phi(x_j^A, x_j^B) \geq 0 \quad \forall \phi \in \Phi,$$

$$\pi_{BC} \in \Pi \;\Leftrightarrow\; \sum_j p_j \phi(x_j^B, x_j^C) \geq 0 \quad \forall \phi \in \Phi,$$

$$\pi_{CA} \in \Pi \;\Leftrightarrow\; \sum_j p_j \phi(x_j^C, x_j^A) \geq 0 \quad \forall \phi \in \Phi.$$

To express the inequalities above in matrix form, let $p$ denote the column vector of state probabilities, with transpose

$$p^\top = \begin{bmatrix} p_1 & p_2 & \ldots & p_{27} \end{bmatrix}.$$

Let $F_\phi$ denote the coefficient matrix associated with $\phi \in \Phi$. Since each $\phi$ is skew-symmetric, $F_\phi$ is fully determined by three parameters:

$$\phi(x,z) = \phi_{xz}, \qquad \phi(x,y) = \phi_{xy}, \qquad \phi(y,z) = \phi_{yz}.$$

The inequality $F_\phi p \geq 0$ generates a cycle for $\phi$ $(A \succsim B \succsim C \succsim A)$, unless $F_\phi p = 0$.

$$F_\phi = \begin{bmatrix} 0 & 0 & 0 & \phi_{xy} & \phi_{xy} & \phi_{xy} & \cdots & -\phi_{xy} & -\phi_{xy} & -\phi_{xy} & \cdots & 0 & 0 & 0 \\ 0 & \phi_{xy} & \phi_{xz} & -\phi_{xy} & 0 & \phi_{yz} & \cdots & 0 & \phi_{xy} & \phi_{xz} & \cdots & -\phi_{xz} & -\phi_{yz} & 0 \\ 0 & -\phi_{xy} & -\phi_{xz} & 0 & -\phi_{xy} & -\phi_{xz} & \cdots & \phi_{xy} & 0 & -\phi_{yz} & \cdots & \phi_{xz} & \phi_{yz} & 0 \end{bmatrix}$$

A violation of strong transitivity requires the existence of a probability vector $p$ such that, for all $\phi \in \Phi$,

$$\sum_j p_j \phi(x_j^A, x_j^B) \geq 0, \qquad \sum_j p_j \phi(x_j^B, x_j^C) \geq 0,$$

but for some $\phi \in \Phi$,

$$\sum_j p_j \phi(x_j^C, x_j^A) > 0.$$

Construct a matrix $F$ that collects the coefficients from all inequalities and includes every $\phi \in \Phi$. The first three rows correspond to the $\phi$ generating the cycle, and for each remaining $\phi' \in (\Phi \setminus \phi)$, the first two rows of $F_{\phi'}$ are appended to $F$. If there exists a probability vector $p$ such that

$$Fp \geq \begin{bmatrix} 0 & 0 & 0^+ & 0 & 0 & \ldots & 0 \end{bmatrix}^\top, \quad p \geq 0, \quad \mathbf{1}^\top p = 1,$$

then strong transitivity is violated.

Let $m$ denote the number of functions $\phi$. Define the primal and dual systems as follows:

**Primal system:**

$$\mathcal{P} = \left\{ p \in \mathbb{R}^{27} \;\middle|\; Fp \geq b,\ p \geq 0,\ \mathbf{1}^\top p = 1 \right\}$$

**Dual (alternative system):**

$$\mathcal{D} = \left\{ (q, \lambda) \in \mathbb{R}^{2m+1} \times \mathbb{R} \;\middle|\; q \geq 0,\ F^\top q + \lambda \mathbf{1} \leq 0,\ b^\top q + \lambda > 0 \right\}$$

By Farkas' lemma, exactly one of the two systems $\mathcal{P}$ or $\mathcal{D}$ has a solution, but not both. If the dual system $\mathcal{D}$ does not have a solution, then the primal system $\mathcal{P}$ admits one, and the strong transitivity is violated.

If $\phi^1$ is the function that generates the cycle and violates strong transitivity, the matrix $F$ is composed of $F_\phi$ and the first two rows of $F_\phi$ for all $\phi \in (\Phi \setminus \phi^1)$. The structure of $F_\phi$ depends on the nature of the states with three outcomes in $X$. There are three distinct cases to consider. First, if all options yield the same outcome, the corresponding column in $F_\phi$ consists entirely of zeros, reflecting the triviality of such states. Second, if only two options share the same outcome, each $\phi$ produces three inequalities, represented as three rows in $F_\phi$, where one element in the column is zero and the remaining two are negatives of each other. It is important to note that, in this case, since all possible states are considered, there must also exist a column in $F_\phi$ that is exactly the negative of this one. This structural property implies that for every such configuration, its inverse configuration is necessarily included. Third, if each option yields a unique outcome, the column associated with each $\phi$ contains all three elements: $\phi(x,z)$, $-\phi(x,y)$, and $-\phi(y,z)$.

$$F = \begin{bmatrix} 0 & 0 & 0 & \phi^1_{xy} & \phi^1_{xy} & \phi^1_{xy} & \cdots & -\phi^1_{xy} & -\phi^1_{xy} & -\phi^1_{xy} & \cdots & 0 & 0 & 0 \\ 0 & \phi^1_{xy} & \phi^1_{xz} & -\phi^1_{xy} & 0 & \phi^1_{yz} & \cdots & 0 & \phi^1_{xy} & \phi^1_{xz} & \cdots & -\phi^1_{xz} & -\phi^1_{yz} & 0 \\ 0 & -\phi^1_{xy} & -\phi^1_{xz} & 0 & -\phi^1_{xy} & -\phi^1_{xz} & \cdots & \phi^1_{xy} & 0 & -\phi^1_{yz} & \cdots & \phi^1_{xz} & \phi^1_{yz} & 0 \\ 0 & 0 & 0 & \phi^2_{xy} & \phi^2_{xy} & \phi^2_{xy} & \cdots & -\phi^2_{xy} & -\phi^2_{xy} & -\phi^2_{xy} & \cdots & 0 & 0 & 0 \\ 0 & \phi^2_{xy} & \phi^2_{xz} & -\phi^2_{xy} & 0 & \phi^2_{yz} & \cdots & 0 & \phi^2_{xy} & \phi^2_{xz} & \cdots & -\phi^2_{xz} & -\phi^2_{yz} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \phi^m_{xy} & \phi^m_{xy} & \phi^m_{xy} & \cdots & -\phi^m_{xy} & -\phi^m_{xy} & -\phi^m_{xy} & \cdots & 0 & 0 & 0 \\ 0 & \phi^m_{xy} & \phi^m_{xz} & -\phi^m_{xy} & 0 & \phi^m_{yz} & \cdots & 0 & \phi^m_{xy} & \phi^m_{xz} & \cdots & -\phi^m_{xz} & -\phi^m_{yz} & 0 \end{bmatrix}$$

To determine whether the system $\mathcal{D}$ admits a solution, it is crucial to analyze the role of the parameter $\lambda$. Let $F^\top$ denote the transpose of $F$. Because $F$ contains columns that are exact negatives of one another, $F^\top$ accordingly possesses rows that are pairwise negatives. As a result, the product $F^\top q$ inherently contains components that are negatives of each other. Consequently, a positive value of $\lambda$ cannot satisfy the system, as adding $\lambda\mathbf{1}$ cannot simultaneously render all elements of $F^\top q + \lambda\mathbf{1}$ smaller than or equal to zero, or in other words, all elements of $F^\top q$ strictly negative. This structural property rules out $\lambda > 0$ as a feasible parameter and therefore directs attention to the case $\lambda \leq 0$ when exploring the existence of potential solutions.

For $\lambda = 0$, the system $\mathcal{D}$ becomes

$$(q, 0): \quad F^\top q \leq 0, \qquad q \geq 0, \qquad q \neq 0, \qquad b^\top q > 0.$$

The only non-zero element of $b$ is its third element. To ensure that $b^\top q > 0$, the third element of $q$ must also be positive and non-zero. Furthermore, each element of $F^\top q$ is smaller than or equal to zero. For those rows of $F^\top$ whose negatives are also present in $F^\top$, the corresponding components of $F^\top q$ must be equal to zero.

$$F^\top = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \phi^1_{xy} & -\phi^1_{xy} & 0 & \phi^2_{xy} & \cdots & 0 & \phi^m_{xy} \\
0 & \phi^1_{xz} & -\phi^1_{xz} & 0 & \phi^2_{xz} & \cdots & 0 & \phi^m_{xz} \\
\phi^1_{xy} & -\phi^1_{xy} & 0 & \phi^2_{xy} & -\phi^2_{xy} & \cdots & \phi^m_{xy} & -\phi^m_{xy} \\
\phi^1_{xy} & 0 & -\phi^1_{xy} & \phi^2_{xy} & 0 & \cdots & \phi^m_{xy} & 0 \\
\phi^1_{xy} & \phi^1_{yz} & -\phi^1_{xz} & \phi^2_{xy} & \phi^2_{yz} & \cdots & \phi^m_{xy} & \phi^m_{yz} \\
\phi^1_{xz} & -\phi^1_{xz} & 0 & \phi^2_{xz} & -\phi^2_{xz} & \cdots & \phi^m_{xz} & -\phi^m_{xz} \\
\phi^1_{xz} & -\phi^1_{yz} & -\phi^1_{xy} & \phi^2_{xz} & -\phi^2_{yz} & \cdots & \phi^m_{xz} & -\phi^m_{yz} \\
\phi^1_{xz} & 0 & -\phi^1_{xz} & \phi^2_{xz} & 0 & \cdots & \phi^m_{xz} & 0 \\
-\phi^1_{xy} & 0 & \phi^1_{xy} & -\phi^2_{xy} & 0 & \cdots & -\phi^m_{xy} & 0 \\
-\phi^1_{xy} & \phi^1_{xy} & 0 & -\phi^2_{xy} & \phi^2_{xy} & \cdots & -\phi^m_{xy} & \phi^m_{xy} \\
-\phi^1_{xy} & \phi^1_{xz} & -\phi^1_{yz} & -\phi^2_{xy} & \phi^2_{xz} & \cdots & -\phi^m_{xy} & \phi^m_{xz} \\
0 & -\phi^1_{xy} & \phi^1_{xy} & 0 & -\phi^2_{xy} & \cdots & 0 & -\phi^m_{xy} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \phi^1_{yz} & -\phi^1_{yz} & 0 & \phi^2_{yz} & \cdots & 0 & \phi^m_{yz} \\
\phi^1_{yz} & -\phi^1_{xz} & \phi^1_{xy} & \phi^2_{yz} & -\phi^2_{xz} & \cdots & \phi^m_{yz} & -\phi^m_{xz} \\
\phi^1_{yz} & -\phi^1_{yz} & 0 & \phi^2_{yz} & -\phi^2_{yz} & \cdots & \phi^m_{yz} & -\phi^m_{yz} \\
\phi^1_{yz} & 0 & -\phi^1_{yz} & \phi^2_{yz} & 0 & \cdots & \phi^m_{yz} & 0 \\
-\phi^1_{xz} & 0 & \phi^1_{xz} & -\phi^2_{xz} & 0 & \cdots & -\phi^m_{xz} & 0 \\
-\phi^1_{xz} & \phi^1_{xy} & \phi^1_{yz} & -\phi^2_{xz} & \phi^2_{xy} & \cdots & -\phi^m_{xz} & \phi^m_{xy} \\
-\phi^1_{xz} & \phi^1_{xz} & 0 & -\phi^2_{xz} & \phi^2_{xz} & \cdots & -\phi^m_{xz} & \phi^m_{xz} \\
-\phi^1_{yz} & -\phi^1_{xy} & \phi^1_{xz} & -\phi^2_{yz} & -\phi^2_{xy} & \cdots & -\phi^m_{yz} & -\phi^m_{xy} \\
-\phi^1_{yz} & 0 & \phi^1_{yz} & -\phi^2_{yz} & 0 & \cdots & -\phi^m_{yz} & 0 \\
-\phi^1_{yz} & \phi^1_{yz} & 0 & -\phi^2_{yz} & \phi^2_{yz} & \cdots & -\phi^m_{yz} & \phi^m_{yz} \\
0 & -\phi^1_{xz} & \phi^1_{xz} & 0 & -\phi^2_{xz} & \cdots & 0 & -\phi^m_{xz} \\
0 & -\phi^1_{yz} & \phi^1_{yz} & 0 & -\phi^2_{yz} & \cdots & 0 & -\phi^m_{yz} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}$$

To simplify the problem, note from the structure of $F$ that

$$\begin{array}{lll}
\text{Row } 10 = -\text{Row } 5, & \text{Row } 11 = -\text{Row } 4, & \text{Row } 13 = -\text{Row } 2, \\
\text{Row } 19 = -\text{Row } 9, & \text{Row } 21 = -\text{Row } 7, & \text{Row } 25 = -\text{Row } 3, \\
\text{Row } 23 = -\text{Row } 18, & \text{Row } 24 = -\text{Row } 17, & \text{Row } 26 = -\text{Row } 15.
\end{array}$$

This implies that the corresponding components of $F^\top q$ must be zero for the rows listed above. Row 8 is equal to Row $5 +$ Row $9 -$ Row 6 and again, since both Row 8 and its negative, Row 6, are in the rows, to satisfy $F^\top q \leq 0$, the corresponding element of $F^\top q$ to those rows must be equal to zero. The corresponding components of $F^\top q$ for rows 12, 16, 20, and 22 must also be zero:

$$\begin{aligned}
\text{Row } 12 &= -\text{Row } 6 + \text{Row } 3 + \text{Row } 15,\\
\text{Row } 16 &= -\text{Row } 8 + \text{Row } 7 + \text{Row } 17,\\
\text{Row } 20 &= -\text{Row } 8 + \text{Row } 13 + \text{Row } 26,\\
\text{Row } 22 &= -\text{Row } 6 + \text{Row } 4 + \text{Row } 24.
\end{aligned}$$

To satisfy $F^\top q \leq 0$, note that, given the structure of $F$, every component of $F^\top q$ must in fact be zero. Moreover, since some rows are repeated, I can further simplify $F$.

$$\begin{aligned}
\text{Row } 5 &= \text{Row } 2 + \text{Row } 4,\\
\text{Row } 9 &= \text{Row } 3 + \text{Row } 7,\\
\text{Row } 18 &= \text{Row } 15 + \text{Row } 17.
\end{aligned}$$

Given that some rows are linear combinations of others, and now that the problem is reduced to $F^\top q = 0$, I can remove those dependent rows. It suffices to keep rows $2, 3, 4, 6, 7, 15, 17$, which I denote in a different order as $F^{\top\star}$:

$$F^{\top\star} = \begin{bmatrix}
0 & \phi^1_{xy} & -\phi^1_{xy} & 0 & \phi^2_{xy} & \cdots & 0 & \phi^m_{xy}\\
0 & \phi^1_{xz} & -\phi^1_{xz} & 0 & \phi^2_{xz} & \cdots & 0 & \phi^m_{xz}\\
0 & \phi^1_{yz} & -\phi^1_{yz} & 0 & \phi^2_{yz} & \cdots & 0 & \phi^m_{yz}\\
\phi^1_{xy} & -\phi^1_{xy} & 0 & \phi^2_{xy} & -\phi^2_{xy} & \cdots & \phi^m_{xy} & -\phi^m_{xy}\\
\phi^1_{xz} & -\phi^1_{xz} & 0 & \phi^2_{xz} & -\phi^2_{xz} & \cdots & \phi^m_{xz} & -\phi^m_{xz}\\
\phi^1_{yz} & -\phi^1_{yz} & 0 & \phi^2_{yz} & -\phi^2_{yz} & \cdots & \phi^m_{yz} & -\phi^m_{yz}\\
\phi^1_{xy} & \phi^1_{yz} & -\phi^1_{xz} & \phi^2_{xy} & \phi^2_{yz} & \cdots & \phi^m_{xy} & \phi^m_{yz}
\end{bmatrix}$$

As mentioned before, the third component of $q$ is strictly positive, $q_{13} > 0$, while all other components are non-negative. Let

$$q^\top = \begin{bmatrix} q_{11} & q_{12} & 1 & q_{21} & q_{22} & \ldots & q_{m1} & q_{m2} \end{bmatrix},$$

$$\begin{aligned}
q_1^\top &= \begin{bmatrix} q_{11} & q_{21} & \ldots & q_{m1} \end{bmatrix},\\
q_2^\top &= \begin{bmatrix} q_{12} & q_{22} & \ldots & q_{m2} \end{bmatrix}.
\end{aligned}$$

Define vectors $\boldsymbol{\phi}_{xz}$, $\boldsymbol{\phi}_{xy}$, and $\boldsymbol{\phi}_{yz}$ as follows:

$$\boldsymbol{\phi}_{xy}^\top = \begin{bmatrix} \phi^1_{xy} & \phi^2_{xy} & \ldots & \phi^m_{xy} \end{bmatrix},$$

$$\boldsymbol{\phi}_{xz}^\top = \begin{bmatrix} \phi_{xz}^1 & \phi_{xz}^2 & \dots & \phi_{xz}^m \end{bmatrix},$$
$$\boldsymbol{\phi}_{yz}^\top = \begin{bmatrix} \phi_{yz}^1 & \phi_{yz}^2 & \dots & \phi_{yz}^m \end{bmatrix}.$$

The equations in $F^{\top\star} q = 0$ can be rewritten as a set of separate equations:

$$q_2^\top \boldsymbol{\phi}_{xy} - \phi_{xy}^1 = 0,$$
$$q_2^\top \boldsymbol{\phi}_{xz} - \phi_{xz}^1 = 0,$$
$$q_2^\top \boldsymbol{\phi}_{yz} - \phi_{yz}^1 = 0,$$
$$q_1^\top \boldsymbol{\phi}_{xy} - q_2^\top \boldsymbol{\phi}_{xy} = 0,$$
$$q_1^\top \boldsymbol{\phi}_{xz} - q_2^\top \boldsymbol{\phi}_{xz} = 0,$$
$$q_1^\top \boldsymbol{\phi}_{yz} - q_2^\top \boldsymbol{\phi}_{yz} = 0,$$
$$q_1^\top \boldsymbol{\phi}_{xy} + q_2^\top \boldsymbol{\phi}_{yz} - \phi_{xz}^1 = 0.$$

By solving these equations, I obtain

$$q_2^\top \boldsymbol{\phi}_{yz} = \phi_{yz}^1,$$
$$q_1^\top \boldsymbol{\phi}_{xy} = q_2^\top \boldsymbol{\phi}_{xy} = \phi_{xy}^1,$$
$$\phi_{xy}^1 + \phi_{yz}^1 - \phi_{xz}^1 = 0.$$

This final equation contradicts the initial assumption that, for $\phi^1$, $\phi_{xy}^1 + \phi_{yz}^1 \neq \phi_{xz}^1$. Therefore, if $\lambda = 0$, the dual system does not have a solution.

If $\lambda < 0$, the system $\mathcal{D}$ becomes

$$(q, \lambda): \quad \max(F^\top q) \leq -\lambda < b^\top q, \qquad q \geq 0, \qquad \lambda < 0.$$

The only non-zero element of $b$ is its third element, $b_3 \to 0^+$. To ensure that $b^\top q > 0$, the third element of $q$ must also be positive and non-zero, $q_{13} > 0$. Furthermore, among the components of $F^\top q$, the maximum must be smaller than or equal to $-\lambda$. Since there are rows in $F^\top$ whose negatives are also present in $F^\top$, $\max(F^\top q)$ is greater than or equal to zero. $\max(F^\top q) = 0$ implies that all components of $F^\top q$ are equal to zero. This leads to a situation analogous to the case I analyzed for $\lambda = 0$, which ultimately results in $\phi_{xy}^1 + \phi_{yz}^1 = \phi_{xz}^1$, thereby contradicting our initial assumption.

$$(q,\lambda): \quad 0 < \max(F^\top q) \leq -\lambda < b^\top q, \qquad q \geq 0, \qquad \lambda < 0,$$

$$q^\top = \begin{bmatrix} q_{11} & q_{12} & 1 & q_{21} & q_{22} & \dots & q_{m1} & q_{m2} \end{bmatrix}.$$

For any arbitrary $b_3 \to 0^+$, I have $0 < max(F^\top q) < b_3$. This implies that each component of $F^\top q$ is strictly less than $b_3$. Accordingly, the independent equations can be rewritten separately as follows:

$$|q_2^\top \boldsymbol{\phi}_{xy} - \phi_{xy}^1| < b_3,$$

$$|q_2^\top \boldsymbol{\phi}_{xz} - \phi_{xz}^1| < b_3,$$

$$|q_2^\top \boldsymbol{\phi}_{yz} - \phi_{yz}^1| < b_3,$$

$$|q_1^\top \boldsymbol{\phi}_{xy} - q_2^\top \boldsymbol{\phi}_{xy}| < b_3,$$

$$|q_1^\top \boldsymbol{\phi}_{xz} - q_2^\top \boldsymbol{\phi}_{xz}| < b_3,$$

$$|q_1^\top \boldsymbol{\phi}_{yz} - q_2^\top \boldsymbol{\phi}_{yz}| < b_3,$$

$$q_1^\top \boldsymbol{\phi}_{xy} + q_2^\top \boldsymbol{\phi}_{yz} - \phi_{xz}^1 < b_3,$$

$$q_1^\top \boldsymbol{\phi}_{xz} - q_2^\top \boldsymbol{\phi}_{yz} - \phi_{xy}^1 < b_3,$$

$$-q_1^\top \boldsymbol{\phi}_{xy} + q_2^\top \boldsymbol{\phi}_{xz} - \phi_{yz}^1 < b_3,$$

$$q_1^\top \boldsymbol{\phi}_{yz} - q_2^\top \boldsymbol{\phi}_{xz} + \phi_{xy}^1 < b_3,$$

$$-q_1^\top \boldsymbol{\phi}_{xz} + q_2^\top \boldsymbol{\phi}_{xy} + \phi_{yz}^1 < b_3,$$

$$-q_1^\top \boldsymbol{\phi}_{yz} - q_2^\top \boldsymbol{\phi}_{xy} + \phi_{xz}^1 < b_3.$$

Consider two of the inequalities:

$$q_1^\top \boldsymbol{\phi}_{xy} + q_2^\top \boldsymbol{\phi}_{yz} - \phi_{xz}^1 < b_3,$$

$$q_1^\top \boldsymbol{\phi}_{xz} - q_2^\top \boldsymbol{\phi}_{yz} - \phi_{xy}^1 < b_3.$$

For each inequality, the left-hand side can be manipulated by adding and subtracting the same term, as shown below:

$$q_1^\top \boldsymbol{\phi}_{xy} + q_2^\top \boldsymbol{\phi}_{yz} - \phi_{xz}^1 =$$

$$q_1^\top \boldsymbol{\phi}_{xy} - q_2^\top \boldsymbol{\phi}_{xy} + q_2^\top \boldsymbol{\phi}_{xy} - \phi^1_{xy} + \phi^1_{xy} + q_2^\top \boldsymbol{\phi}_{yz} - \phi^1_{yz} + \phi^1_{yz} - \phi^1_{xz} =$$

$$(q_1^\top \boldsymbol{\phi}_{xy} - q_2^\top \boldsymbol{\phi}_{xy}) + (q_2^\top \boldsymbol{\phi}_{xy} - \phi^1_{xy}) + (q_2^\top \boldsymbol{\phi}_{yz} - \phi^1_{yz}) + (\phi^1_{xy} + \phi^1_{yz} - \phi^1_{xz}),$$

$$q_1^\top \boldsymbol{\phi}_{xz} - q_2^\top \boldsymbol{\phi}_{yz} - \phi^1_{xy} =$$

$$q_1^\top \boldsymbol{\phi}_{xz} - q_2^\top \boldsymbol{\phi}_{xz} + q_2^\top \boldsymbol{\phi}_{xz} - \phi^1_{xz} + \phi^1_{xz} - q_2^\top \boldsymbol{\phi}_{yz} + \phi^1_{yz} - \phi^1_{yz} - \phi^1_{xy} =$$

$$(q_1^\top \boldsymbol{\phi}_{xz} - q_2^\top \boldsymbol{\phi}_{xz}) + (q_2^\top \boldsymbol{\phi}_{xz} - \phi^1_{xz}) - (q_2^\top \boldsymbol{\phi}_{yz} - \phi^1_{yz}) - (\phi^1_{xy} + \phi^1_{yz} - \phi^1_{xz}).$$

Taking the terms in the first three parentheses to the right-hand side of each inequality, I obtain:

$$(\phi^1_{xy} + \phi^1_{yz} - \phi^1_{xz}) < b_3 - (q_1^\top \boldsymbol{\phi}_{xy} - q_2^\top \boldsymbol{\phi}_{xy}) - (q_2^\top \boldsymbol{\phi}_{xy} - \phi^1_{xy}) - (q_2^\top \boldsymbol{\phi}_{yz} - \phi^1_{yz}),$$

$$-(\phi^1_{xy} + \phi^1_{yz} - \phi^1_{xz}) < b_3 - (q_1^\top \boldsymbol{\phi}_{xz} - q_2^\top \boldsymbol{\phi}_{xz}) - (q_2^\top \boldsymbol{\phi}_{xz} - \phi^1_{xz}) + (q_2^\top \boldsymbol{\phi}_{yz} - \phi^1_{yz}).$$

According to the condition $\max(F^\top q) < b_3$, each of the expressions in the parentheses on the right-hand side of each inequality has an absolute value strictly less than $b_3$. Substituting these bounds into the inequalities,

$$(\phi^1_{xy} + \phi^1_{yz} - \phi^1_{xz}) < 4b_3,$$

$$-(\phi^1_{xy} + \phi^1_{yz} - \phi^1_{xz}) < 4b_3.$$

By the initial assumption $\phi^1_{xy} + \phi^1_{yz} - \phi^1_{xz} \neq 0$, the bounds $\max(F^\top q) < b_3$ imply the necessary feasibility condition

$$|\phi^1_{xy} + \phi^1_{yz} - \phi^1_{xz}| < 4b_3$$

for the dual system $\mathcal{D}$. Hence, choosing $b_3 > 0$ so small that

$$b_3 < \frac{1}{4}|\phi^1_{xy} + \phi^1_{yz} - \phi^1_{xz}|$$

makes $\mathcal{D}$ infeasible for $\lambda < 0$. By Farkas' lemma, $\mathcal{P}$ is then feasible. Consequently, if there exists $\phi \in \Phi$ and at least three distinct outcomes in $X$ such that $\phi(x, z) \neq \phi(x, y) + \phi(y, z)$, some $b_3 > 0$ can be selected so that the primal system admits a solution that constitutes a cycle and violates strong transitivity.

It has been shown previously that for $\lambda \geq 0$, no $q$ exists such that the dual system has a solution; therefore, the primal system admits a solution. Based on this, I can argue that the existence of at least three elements $x, y, z \in X$ for which

$$\phi(x, z) \neq \phi(x, y) + \phi(y, z)$$

for at least one function $\phi \in \Phi$, leads to a violation of strong transitivity. Hence, the condition

$$\phi(x, z) = \phi(x, y) + \phi(y, z) \quad \text{for all } \phi \in \Phi$$

is necessary for strong transitivity.